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RESEARCH
REPORT
N° 8341
August 2013
Project-Team S4

ISSN 0249-6399   ISRN INRIA/RR-8341--FR+ENG

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Research Report n° 8341 — August 2013 — 54 pages

Abstract: We introduce a two-sorted substructural logic called 'Update Logic' where the central objects of study are updates, which are represented formally by ternary relations. We develop a basic correspondence theory which relates properties of ternary relations with axioms and inference rules stating properties of updates. We claim that update logic can capture various logic-based formalisms dealing with belief change. As case study, we consider the logical framework of Dynamic Epistemic Logic (DEL) and we show that we can embed it within update logic. Also, we identify axioms and inference rules that completely characterize the DEL product update. Moreover, we introduce Gentzen calculi which extend Gentzen calculi for modal logic and which axiomatize our update logic and DEL. Our completeness proof techniques are new compared to the standard proof techniques used to prove completeness of Gentzen calculi. Our contributions to proof theory are independent from our contributions to the study of logical dynamics and can also be read independently.

Key-words: Substructural logics, dynamic epistemic logic, update, Gentzen system.

The results of this report are new. Parts of Sections 2, 3.2 and 5.4 will soon be published in (Aucher, 2013) with some minor changes. Also, a part of the introduction of (Aucher, 2011) is reproduced with minor changes in the beginning of Section 5.4.2.

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Logique de la Mise à Jour

Résumé : Nous introduisons une logique substructurelle typée appelée 'Logique de la Mise à Jour' où les objets d'étude sont les mises à jour, qui sont représentées formellement par des relations ternaires. Nous développons une théorie élémentaire de la correspondance qui relie les propriétés des relations ternaires avec des axiomes et des règles d'inférence qui énoncent des propriétés des mises à jour. Nous soutenons que notre logique de la mise à jour peut capturer de nombreux formalismes logiques qui ont pour objet d'étude le changement de croyances. Comme cas d'étude, nous considérons le formalisme de la logique épistémique dynamique (DEL) et nous montrons que nous pouvons l'intégrer dans la logique de la mise à jour. Par ailleurs, nous identifions des axiomes et des règles d'inférence qui caractérisent complétement le produit de mise à jour de DEL. De plus, nous introduisons deux calculs de séquents qui étendent les calculs de séquents pour la logique modale et qui axiomatisent notre logique de la mise à jour et DEL. Nos techniques de preuve de complétude sont nouvelles comparées aux techniques de preuve standards utilisées pour prouver la complétude des calculs de séquents. Nos contributions à la théorie de la preuve sont indépendantes de nos contributions à l'étude du dynamisme logique et peuvent être lues de manière indépendantes.

Mots-clés : Logiques substructurelles, logique épistémique dynamique, mise à jour, calcul des séquents.

Contents

1 Introduction

In everyday life, the way we update and revise our beliefs plays an important role in our representation of the surrounding world and therefore also in our decision making process. This has lead researchers in artificial intelligence and computer science to develop logic-based theories that study and formalize belief change and the so-called "common sense reasoning". The rationale underlying the development of such theories is that it would ultimately help us understand our everyday life reasoning and the way we update our beliefs, and that the resulting work could subsequently lead to the development of tools that could be used for example by artificial agents in order to act autonomously in an uncertain and changing world. A number of theories have been proposed to capture different kinds of updates and the reasoning styles that they induce, using different formalisms and under various assumptions: dynamic epistemic logic (van Benthem, 2011a; van Ditmarsch et al., 2007), default and non-monotonic logics (Makinson, 2005; Gabbay et al., 1998), belief revision theory (Gärdenfors, 1988), conditional logic (Nute and Cross, 2001), etc... However, a generic and general framework encompassing all these theories is still lacking. Instead, the current state of the art is such that we are left with various formalisms which are difficult to relate formally to each other despite numerous attempts (Gärdenfors, 1991; Makinson and Gärdenfors, 1989; Aucher, 2010; van Ditmarsch et al., 2004; Aucher, 2004; Baltag and Smets, 2008b), partly because they rely on different kinds of formalisms. Our goal in this report is to propose a logical framework based on the very general framework of substructural logics, where updates are the central objects of study, and that is abstract enough to be able to embed the existing dynamic theories of belief change.

Our proposal will be based on the key observation that an update can be represented abstractly as a ternary relation: the first argument of the ternary relation represents the initial situation/state, the second the event that occurs in this initial situation (the informative input) and the third the resulting situation/state after the occurrence of the event. On the basis of this observation, we will then define a logic called *update logic*. Update logic is a substructural logic: the semantics of substructural logics also relies on a ternary relation introduced by Routley and Meyer (1972a; 1972b; 1973; 1982) for relevance logic in the 1970's. In substructural logics, the introduction of this relation was originally motivated by technical reasons (any two-ary connective can be given a semantics in terms of a ternary relation), and providing a non-circular and conceptually grounded interpretation of this relation remains problematic (Beall et al., 2012). As we shall see, our dynamic interpretation of ternary relations as updates provides a conceptual foundation for the Routley and Meyer's semantics which is also consistent with some of the interpretations of this ternary relation proposed in the literature.

As case study, we will consider throughout this report the logical framework of Dynamic Epistemic Logic (DEL) that we will embed within our update logic. We leave the embedding of the other logic-based theories of belief change within update logic for future work. DEL is an influential logical framework for reasoning about the dynamics of beliefs and knowledge which has drawn the attention of a number of researchers ever since the seminal publication of (Batlag et al., 1998). In fact, DEL has sometimes been called "update logic". A number of contributions have linked DEL to older and more established logical frameworks: it has been embedded into (automata) PDL (van Eijck, 2004; van Benthem and Kooi, 2004), it has been given an algebraic semantics (Baltag et al., 2005, 2007), and it has been related to epistemic temporal logic (van Benthem et al., 2009a; Aucher and Herzig, 2011) and the situation calculus (van Benthem, 2011b; van Ditmarsch et al., 2009). Despite these connections, DEL remains, arguably, a rather isolated logic in the vast realm of non-classical logics and modal logics. This is problematic if logic is to be viewed ultimately as a unified and unifying field and if we want to avoid that DEL goes on "riding off madly in all directions" (a metaphor used by van Benthem (2011a) about logic in general). We will show that DEL can be embedded within our update logic. This entails that DEL is a substructural logic. In particular, we will show that the operators of progression, regression and epistemic planning introduced in (Aucher, 2011, 2012) correspond formally to standard operators of substructural logics (like the Lambek Calculus). To demonstrate the generic character of our update logic, we will sketch a correspondence theory and we will provide axioms and inference rules that completely characterize the DEL product update.

From a technical point of view, our main contribution in this report is to develop a basic correspondence theory for update logic and to propose a new method for proving completeness of Gentzen calculi. This new method relies on an adaptation of the classical Henkin construction that builds models. As a result of this new approach, we obtain Gentzen calculi for update logic and its DEL extension which are mere extensions of the standard Gentzen calculi for propositional and modal logic.

The report is organized as follows. In Section 2, we recall the core of DEL viewed from a semantic perspective. In Section 3, we briefly recall elementary notions of relevance and substructural logics and we observe that the ternary relation of relevance logic (Restall, 2000, 2006; Dunn and Restall, 2002) can be interpreted intuitively as a kind of update. Then, we proceed further to define our update logic based on this idea. In Section 4, we provide a sound and complete Gentzen sequent calculus for our update logic. The method for proving completeness of Gentzen calculi is new and an explanation of the main new ideas is given for the case of modal (and propositional) logic in Section 4.1, which can be read independently from the rest of the report. In Section 5, we consider the DEL product update as case study, and we provide axiom schemata and inference rules based on (Aucher, 2011, 2012) which characterize completely the DEL product update. Also, we provide a Gentzen calculus for DEL. Finally, we show how the different operators of (Aucher, 2011, 2012) correspond formally to standard operators of substructural logics like the Lambek Calculus. We conclude in Section 6.

Note. The results of this report are new. Parts of Sections 2, 3.2 and 5.4 will soon be published in (Aucher, 2013) with some minor changes. Also, a part of the introduction of (Aucher, 2011) is reproduced with minor changes in the beginning of Section 5.4.2.

2 Dynamic Epistemic Logic

Dynamic epistemic logic (DEL) is a relatively recent non-classical logic (Batlag et al., 1998) which extends ordinary modal epistemic logic (Hintikka, 1962) by the inclusion of *event models* (called $\mathcal{L}_\alpha$-models in this report) to describe actions, and a *product update* operator that defines how epistemic models are updated as the consequence of executing actions described through event models (see (Baltag and Moss, 2004; van Ditmarsch et al., 2007; van Benthem, 2011a) for more details). So, the methodology of DEL is such that it splits the task of representing the agents' beliefs and knowledge into three parts: first, one represents their beliefs about an initial situation; second, one represents their beliefs about an event taking place in this situation; third, one represents the way the agents update their beliefs about the situation after (or during) the occurrence of the event. Following this methodology, we also split the exposition of the DEL framework into three sections.

2.1 Representation of the Initial Situation: $\mathcal{L}$-model

In the rest of this report, ATM is a countable set of propositional letters called *atomic facts* which describe static situations, and $AGT := \{1, \dots, m\}$ is a finite set of agents.

Definition 1 (Language $\mathcal{L}$ and $\mathcal{L}$–structure)**.** We define the language $\mathcal{L}$ inductively as follows:

$$\mathcal{L} : \varphi \quad ::= \quad p \quad | \quad \neg\varphi \quad | \quad \varphi \wedge \varphi \quad | \quad \varphi \vee \varphi \quad | \quad \Box_j \varphi$$

where p ranges over ATM and j over AGT. We define $\bot := p \wedge \neg p$ for a chosen $p \in ATM$ and we also define $\top := \neg\bot$. The formula $\Diamond_j \varphi$ is an abbreviation for $\neg\Box_j\neg\varphi$, the formula $\varphi \to \psi$ is an abbreviation for $\neg\varphi \vee \psi$, and the formula $\varphi \leftrightarrow \psi$ is an abbreviation for $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

A $\mathcal{L}$-*structure* is defined inductively as follows, with φ ranging over $\mathcal{L}$:

$$\mathcal{S} : X \quad ::= \quad \varphi \quad | \quad X, X$$

We abusively write $\varphi \in X$ when the formula $\varphi \in \mathcal{L}$ is a substructure of X. □

A (pointed) $\mathcal{L}$–model $(\mathcal{M}, w)$ represents how the actual world represented by w is perceived by the agents. Atomic facts are used to state properties of this actual world.

Definition 2 ($\mathcal{L}$-model)**.** A $\mathcal{L}$-*model* is a tuple $\mathcal{M} = (W, R_1, \dots, R_m, I)$ where:

- W is a non-empty set of possible worlds,
- $R_j \subseteq W \times W$ is an accessibility relation on W, for each $j \in AGT$,
- $I : W \to 2^{ATM}$ is a function assigning to each possible world a subset of ATM. The function I is called an *interpretation*.

A $\mathcal{L}$-*frame* is a $\mathcal{L}$-model without interpretation. We write $w \in \mathcal{M}$ for $w \in W$, and $(\mathcal{M}, w)$ is called a pointed $\mathcal{L}$-model (w often represents the actual world). We denote by $\mathcal{C}$ the set of pointed $\mathcal{L}$-models and by $\mathcal{C}^F$ the class of pointed $\mathcal{L}$-frames. If $w, v \in W$, we write wR_jv or $(\mathcal{M}, w)R_j(\mathcal{M}, v)$ for $(w, v) \in R_j$, and $R_j(w)$ denotes $\{v \in W \mid wR_jv\}$. $\square$

Intuitively, wR_jv means that in world w agent j considers that world v might correspond to the actual world. Then, we define the following epistemic language that can be used to describe and state properties of $\mathcal{L}$-models as follows. The formula $\square_j\varphi$ reads as "agent j believes φ". Its truth conditions are defined in such a way that agent j believes φ holds in a possible world when φ holds in all the worlds agent j considers possible.

Definition 3 (Truth conditions of $\mathcal{L}$)**.** Let $\mathcal{M}$ be a $\mathcal{L}$-model, $w \in \mathcal{M}$ and $\varphi \in \mathcal{L}$. $\mathcal{M}, w \models \varphi$ is defined inductively as follows:

$$\begin{array}{lll} \mathcal{M}, w \models p & \text{iff} & p \in I(w) \\ \mathcal{M}, w \models \neg\psi & \text{iff} & \text{not } \mathcal{M}, w \models \varphi \\ \mathcal{M}, w \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\ \mathcal{M}, w \models \varphi \vee \psi & \text{iff} & \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi \\ \mathcal{M}, w \models \square_j\varphi & \text{iff} & \text{for all } v \in R_j(w), \mathcal{M}, v \models \varphi \quad \square \end{array}$$

We extend the scope of the relation $\models$ to also relate pointed $\mathcal{L}$-models to structures:

$$\mathcal{M}, w \models X, Y \quad \text{iff} \quad \mathcal{M}, w \models X \text{ and } \mathcal{M}, w \models Y$$

Let C be a class of pointed $\mathcal{L}$–models, let X, Y be $\mathcal{L}$-structures. We say that X *entails* Y *in the class* C, which we write $X \models_C Y$, when the following holds:

$$X \models_C Y \quad \text{iff} \quad \begin{array}{l} \text{for all pointed } \mathcal{L}\text{-model } (\mathcal{M}, w) \in C, \text{ if for all } \varphi \in X\ \mathcal{M}, w \models \varphi, \\ \text{then there is } \psi \in Y \text{ such that } \mathcal{M}, w \models \psi. \end{array}$$

We also write $X \models Y$ for $X \models_{\mathcal{C}} Y$, where $\mathcal{C}$ is the class of all pointed $\mathcal{L}$–models.

Example 1. Assume that agents A, B and C play a card game with three cards: a white one, a red one and a blue one. Each of them has a single card but they do not know the cards of the other players. At each step of the game, some of the players show their/her/his card to another player or to both other players, either privately or publicly. We want to study and represent the dynamics of the agents' beliefs in this game. The initial situation is represented by the pointed $\mathcal{L}$-model $(\mathcal{M}, w)$ of Figure 1. In this example, $AGT := \{A, B, C\}$ and $ATM := \{r_j, b_j, w_j \mid j \in AGT\}$ where r_j stands for 'agent j has the *r*ed card', b_j stands for 'agent j has the *b*lue card' and w_j stands for 'agent j has the *w*hite card'. The boxed possible world corresponds to the actual world. The propositional letters not mentioned in the possible worlds do not hold in these possible worlds. The accessibility relations are represented by arrows indexed by agents between possible worlds. Reflexive arrows are omitted in the figure, which means that for all worlds $v \in \mathcal{M}$ and all agents $j \in AGT$, $v \in R_j(v)$. In this model, we have for example the following statement: $\mathcal{M}, w \models (w_B \wedge \neg\square_A w_B) \wedge \square_C\neg\square_A w_B$. It states that player A does not 'know' that player B has the white card and player C 'knows' it. $\square$

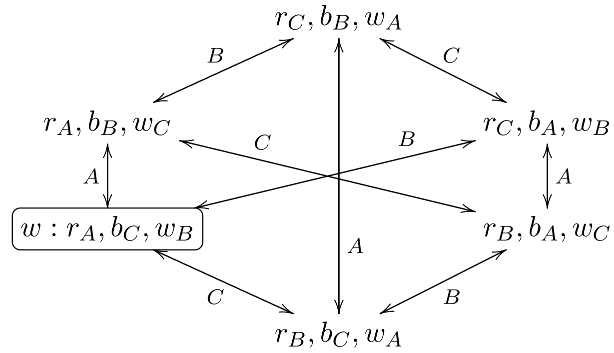

Figure 1: Cards Example

2.2 Representation of the Event: $\mathcal{L}_\alpha$-model

The language $\mathcal{L}_\alpha$ was introduced by Baltag et al. (1999). The propositional letters p_ψ describing events are called *atomic events* and range over $ATM_\alpha = \{p_\psi : \psi \text{ ranges over } \mathcal{L}\}$. The reading of p_ψ is "an event of precondition ψ is occurring".

Definition 4 (Language $\mathcal{L}_\alpha$ and $\mathcal{L}_\alpha$–structure)**.** We define the language $\mathcal{L}_\alpha$ inductively as follows:

$$\mathcal{L}_\alpha : \alpha \quad ::= \quad p_\psi \quad | \quad \neg\alpha \quad | \quad \alpha \wedge \alpha \quad | \quad \alpha \vee \alpha \quad | \quad \Box_j \alpha$$

where ψ ranges over $\mathcal{L}$ and j over AGT. We define $\bot := p_\psi \wedge \neg p_\psi$ for a chosen $\psi \in \mathcal{L}$ and we define $\top := \neg\bot$. The formula $\Diamond_j\alpha$ is an abbreviation for $\neg\Box_j\neg\alpha$, the formula $\alpha \to \beta$ is an abbreviation for $\neg\alpha \vee \beta$, and the formula $\alpha \leftrightarrow \beta$ is an abbreviation for $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

A $\mathcal{L}_\alpha$-*structure* is defined inductively as follows, with β ranging over $\mathcal{L}_\alpha$:

$$\mathcal{S}^\alpha : \quad X^\alpha \quad ::= \quad \beta \quad | \quad X^\alpha, X^\alpha$$

We abusively write $\alpha \in X^\alpha$ when the formula $\alpha \in \mathcal{L}_\alpha$ is a substructure of X^α. $\Box$

A pointed $\mathcal{L}_\alpha$-model $(\mathcal{E}, e)$ represents how the actual event represented by e is perceived by the agents. Intuitively, $f \in R_j^\alpha(e)$ means that while the possible event represented by e is occurring, agent j considers possible that the possible event represented by f is actually occurring.

Definition 5 ($\mathcal{L}_\alpha$–model, Batlag et al. 1998)**.** A $\mathcal{L}_\alpha$-*model* is a tuple $\mathcal{E} = (W^\alpha, R_1^\alpha, \ldots, R_m^\alpha, I^\alpha)$ where:

- W^α is a non-empty set of possible events,
- $R_j^\alpha \subseteq W^\alpha \times W^\alpha$ is an accessibility relation on W^α, for each $j \in AGT$,

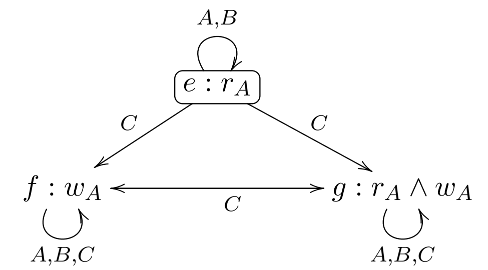

Figure 2: Players A and B show their cards to each other in front of player C

- $I^\alpha : W^\alpha \to \mathcal{L}$ is a function assigning to each possible event a formula of $\mathcal{L}$. The function I^α is called the *precondition function.*

Let P be a subset of $\mathcal{L}$. A *P–complete $\mathcal{L}_\alpha$–model* is a $\mathcal{L}_\alpha$–model which satisfies moreover the following condition:

- $I^\alpha(e) \in P$, for each $e \in W^\alpha$ *(P-complete)*

A *$\mathcal{L}_\alpha$–frame* is a $\mathcal{L}_\alpha$–model without precondition function I^α. We abusively write $e \in \mathcal{E}$ for $e \in W^\alpha$, and $(\mathcal{E}, e)$ is called a *pointed $\mathcal{L}_\alpha$-model* (e often represents the actual event). We denote by $\mathcal{C}^\alpha$ the set of pointed $\mathcal{L}_\alpha$-models, by $\mathcal{C}^\alpha_P$ the set of pointed P-complete event models and by $\mathcal{C}^F_\alpha$ the class of pointed $\mathcal{L}_\alpha$-frames. If $e, f \in W^\alpha$, we write $eR^\alpha_j f$ or $(\mathcal{E}, e)R^\alpha_j(\mathcal{E}, f)$ for $(e, f) \in R^\alpha_j$, and $R^\alpha_j(e)$ denotes $\{f \in W^\alpha \mid eR^\alpha_j f\}$. $\square$

The truth conditions of the language $\mathcal{L}_\alpha$ are identical to the truth conditions of the language $\mathcal{L}$:

Definition 6 (Truth conditions of $\mathcal{L}_\alpha$)**.** Let $\mathcal{E}$ be a $\mathcal{L}_\alpha$-model, $e \in \mathcal{E}$ and $\alpha \in \mathcal{L}_\alpha$. $\mathcal{E}, e \models \alpha$ is defined inductively as follows:

$$
\begin{array}{lll}
\mathcal{E}, e \models p_\psi & \text{iff} & I^\alpha(e) = \psi \\
\mathcal{E}, e \models \neg\alpha & \text{iff} & \text{not } \mathcal{E}, e \models \alpha \\
\mathcal{E}, e \models \alpha \wedge \beta & \text{iff} & \mathcal{E}, e \models \alpha \text{ and } \mathcal{E}, e \models \beta \\
\mathcal{E}, e \models \alpha \vee \beta & \text{iff} & \mathcal{E}, e \models \alpha \text{ or } \mathcal{E}, e \models \beta \\
\mathcal{E}, e \models \Box_j\alpha & \text{iff} & \text{for all } f \in R^\alpha_j(e), \mathcal{E}, f \models \alpha
\end{array}
$$

Let C be a class of pointed $\mathcal{L}_\alpha$–models, let X^α, Y^α be $\mathcal{L}_\alpha$–structures. We say that X *entails* Y *in the class* C, which we write $X^\alpha \models_C Y^\alpha$, when the following holds:

$$
\begin{array}{lll}
X^\alpha \models_C Y^\alpha & \text{iff} & \text{for all pointed } \mathcal{L}_\alpha\text{-model } (\mathcal{E}, e) \in C, \\
& & \text{if for all } \alpha \in X^\alpha\ \mathcal{E}, e \models \alpha, \text{ then there is } \beta \in Y^\alpha \text{ such that } \mathcal{E}, e \models \beta.
\end{array}
$$

We also write $X^\alpha \models Y^\alpha$ for $X^\alpha \models_{\mathcal{C}^\alpha} Y^\alpha$, where $\mathcal{C}^\alpha$ is the class of all pointed $\mathcal{L}_\alpha$–models. $\square$

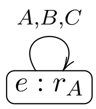

Figure 3: Public announcement of r_A

Example 2. Let us resume Example 1 and assume that players A and B show their card to each other. As it turns out, C noticed that A showed her card to B but did not notice that B did so to A. Players A and B know this. This event is represented in the $\mathcal{L}_\alpha$-model $(\mathcal{E}, e)$ of Figure 2. The boxed possible event e corresponds to the actual event 'player A shows her $r$ed card' (with precondition r_A), f stands for the event 'player A shows her $w$hite card' (with precondition w_A) and g stands for the atomic event 'players A and B show their $r$ed and $w$hite cards respectively to each other' (with precondition $r_A \wedge w_A$).

$$\begin{aligned} \mathcal{E}, e \models\ & p_{r_A} \wedge (\Diamond_A p_{r_A} \wedge \Box_A p_{r_A}) \wedge (\Diamond_B p_{r_A} \wedge \Box_B p_{r_A}) \\ & \wedge (\Diamond_C p_{w_A} \wedge \Diamond_C p_{r_A \wedge w_A} \wedge \Box_C (p_{w_A} \vee p_{r_A \wedge w_A})) \end{aligned} \tag{1}$$

It states that players A and B show their cards to each other, players A and B 'know' this and consider it possible, while player C considers possible that player A shows her $w$hite card and also considers possible that player A shows her $r$ed card, since he does not know her card. In fact, that is all that player C considers possible since he believes that either player A shows her $r$ed card or her $w$hite card.

The $\mathcal{L}_\alpha$-model of Figure 3 corresponds to a 'public announcement' or 'public display' of the fact that agent A has the red card. In particular, the following statement holds in the example of Figure 3:

$$\begin{aligned} \mathcal{E}, e \models\ & p_{r_A} \wedge \Box_A p_{r_A} \wedge \Box_B p_{r_A} \wedge \Box_C p_{r_A} \\ & \wedge \Box_A \Box_A p_{r_A} \wedge \Box_A \Box_B p_{r_A} \wedge \Box_A \Box_C p_{r_A} \\ & \wedge \Box_B \Box_A p_{r_A} \wedge \Box_B \Box_B p_{r_A} \wedge \Box_B \Box_C p_{r_A} \\ & \wedge \Box_C \Box_A p_{r_A} \wedge \Box_C \Box_B p_{r_A} \wedge \Box_C \Box_C p_{r_A} \\ & \wedge \ldots \end{aligned}$$

It states that player A shows her red card and that players A, B and C 'know' it, that players A, B and C 'know' that each of them 'know' it, etc... in other words, there is common knowledge among players A, B and C that player A shows her red card.[1]

$$\mathcal{E}, e \models p_{r_A} \wedge \Box^*_{AGT} p_{r_A}.$$

$\Box$

[1] We write $\mathcal{E}, e \models \Box^*_{AGT} \alpha$ when for all $f \in \left(\bigcup_{j \in AGT} R^\alpha_j \right)^* (e)$, $\mathcal{E}, f \models \alpha$. See for example (Fagin et al., 1995) for a detailed study of the operator $\Box^*_{AGT}$ of common knowledge

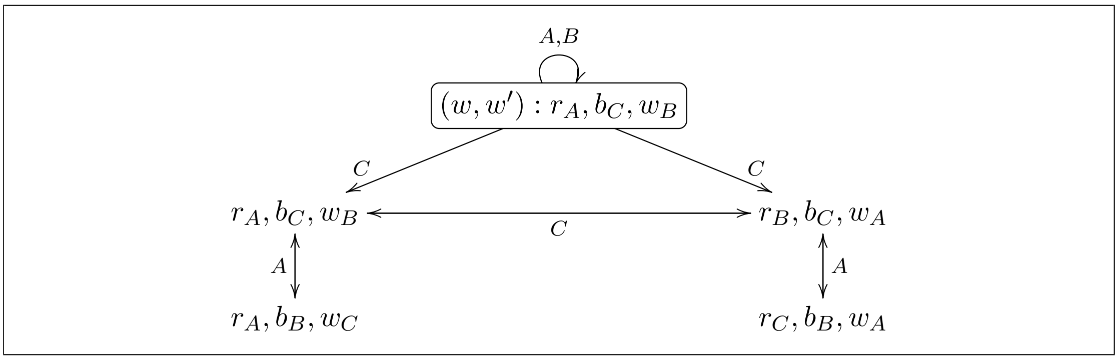

Figure 4: Situation after the Update of the Situation Represented in Figure 1 by the Event Represented in Figure 2

2.3 Update of the Initial Situation by the Event: Product Update

The DEL product update of (Batlag et al., 1998) is defined as follows. This update yields a new $\mathcal{L}$-model $(\mathcal{M}, w) \otimes (\mathcal{E}, e)$ representing how the new situation which was previously represented by $(\mathcal{M}, w)$ is perceived by the agents after the occurrence of the event represented by $(\mathcal{E}, e)$.

Definition 7 (Product update)**.** Let $(\mathcal{M}, w) = (W, R_1, \ldots, R_m, I, w)$ be a pointed $\mathcal{L}$-model and let $(\mathcal{E}, e) = (W^\alpha, R_1^\alpha, \ldots, R_m^\alpha, I, e)$ be a pointed $\mathcal{L}_\alpha$-model such that $\mathcal{M}, w \models I^\alpha(e)$. The *product update of* $(\mathcal{M}, w)$ *and* $(\mathcal{E}, e)$ is the pointed $\mathcal{L}$-model $(\mathcal{M} \otimes \mathcal{E}, (w, e)) = (W^\otimes, R_1^\otimes, \ldots, R_m^\otimes, I^\otimes, (w, e))$ defined as follows: for all $v \in W$ and all $f \in W^\alpha$,

- $W^\otimes = \{(v, f) \in W \times W^\alpha \mid \mathcal{M}, v \models I^\alpha(f)\}$,
- $R_j^\otimes(v, f) = \{(u, g) \in W^\otimes \mid u \in R_j(v) \text{ and } g \in R_j^\alpha(f)\}$,
- $I^\otimes(v, f) = I(v)$. □

Example 3. As a result of the event described in Example 2, the agents update their beliefs. We get the situation represented in the $\mathcal{L}$-model $(\mathcal{M}, w) \otimes (\mathcal{E}, e)$ of Figure 4. In this model, we have for example the following statement:

$$(\mathcal{M}, w) \otimes (\mathcal{E}, e) \models (w_B \wedge B_A w_B) \wedge B_C \neg B_A w_B.$$

It states that player A 'knows' that player B has the white card but player C believes that it is not the case. □

3 Substructural Logics and Update Logic

Substructural logics are a family of logics lacking some of the structural rules of classical logic. A structural Rule is a Rule of inference which is closed under substitution of formulas. The structural rules for classical logic are given in Figure 5. While (*Weakening*)

Weakening:

$$\frac{X \vdash Y}{\varphi, X \vdash Y} W_L \qquad \frac{X \vdash Y}{X \vdash Y, \varphi} W_R$$

Associativity:

$$\frac{X, (Y, Z) \vdash U}{(X, Y), Z \vdash U} B$$

Contraction:

$$\frac{\varphi, \varphi, X \vdash Y}{\varphi, X \vdash Y} C_L \qquad \frac{X \vdash Y, \varphi, \varphi}{X \vdash Y, \varphi} C_R$$

Commutativity:

$$\frac{Y, X \vdash Z}{X, Y \vdash Z} P_L$$

Figure 5: Structural Rules

and (*Contraction*) are often dropped like in relevance logic and linear logic, the Rule of (*Associativity*) is often preserved. We shall see in this report that DEL invalidates all of them.

3.1 Substructural Logics

Our exposition of substructural logics is based on (Restall, 2000, 2006; Dunn and Restall, 2002). The logical framework presented in (Restall, 2000) is much more general and studies a wide range of substructural logics: relevance logic, linear logic, lambek calculus, display logic, etc... For what concerns us in this report, we will only introduce a fragment of this general framework. The semantics of substructural logics is based on the ternary relation of the frame semantics for relevant logic originally introduced by Routley and Meyer (1972a; 1972b; 1973; 1982). Another semantics proposed independently by Urquhart (1971; 1972a; 1972b) at about the same time will be discussed at the end of this section.

Definition 8 (Language $\mathcal{L}_{\mathsf{Sub}}$ and $\mathcal{L}_{\mathsf{Sub}}$–structure)**.** The *language* $\mathcal{L}_{Sub}$ is defined inductively as follows:

$$\begin{array}{rl} \mathcal{L}_{\mathsf{Sub}}: \quad \varphi \ ::= & \top \mid \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \\ & \varphi \supset \varphi \mid \varphi \subset \varphi \mid \varphi \circ \varphi \end{array}$$

where p ranges over ATM.

A $\mathcal{L}_{Sub}$*–structure* is defined inductively as follows, with φ ranging over $\mathcal{L}_{\mathsf{Sub}}$:

$$X \ ::= \ \varphi \mid (X, X) \mid (X; X)$$ □

Definition 9 (Point set, plump accessibility relation)**.** A *point set* $\mathcal{P} = (P, \sqsubseteq)$ is a set P together with a partial order $\sqsubseteq$ on P. The set $Prop(\mathcal{P})$ of *propositions* on $\mathcal{P}$ is the set of all subsets X of P which are *closed upwards*: that is, if $x \in X$ and $x \sqsubseteq x'$ then $x' \in X$. When $\sqsubseteq$ is the identity relation, we say that $\mathcal{P}$ is *flat.* We abusively write $x \in \mathcal{P}$ for $x \in P$.

- A binary relation $\mathcal{T}$ is a *positive two–place accessibility relation* on the point set $\mathcal{P}$ iff for any $x, y \in \mathcal{P}$ where $x\mathcal{T}y$, if $x' \sqsubseteq x$ then there is a $y' \sqsupseteq y$, where $x'\mathcal{T}y'$. Similarly, if $x\mathcal{T}y$ and $y \sqsubseteq y'$ then there is some $x' \sqsubseteq x$, where $x'\mathcal{T}y'$.

- A ternary relation $\mathcal{R}$ is a *three–place accessibility relation* iff whenever $\mathcal{R}xyz$ and $z \sqsubseteq z'$ then there are $y' \sqsupseteq y$ and $x' \sqsubseteq x$, where $\mathcal{R}x'y'z'$. Similarly, if $x' \sqsubseteq x$ then there are $y' \sqsubseteq y$ and $z' \sqsupseteq z$, where $\mathcal{R}x'y'z'$, and if $y' \sqsubseteq y$ then there are $x' \sqsubseteq x$ and $z' \sqsupseteq z$, where $\mathcal{R}x'y'z'$.

- A ternary relation $\mathcal{R}$ is a *plump accessibility relation* on the point set $\mathcal{P}$ if and only if for any $x, y, z, x', y', z' \in \mathcal{P}$ such that $\mathcal{R}xyz$, if $x' \sqsubseteq x$, $y' \sqsubseteq y$ and $z \sqsubseteq z'$, then $\mathcal{R}x'y'z'$. $\square$

Note that the conditions satisfied by $\sqsubseteq$ in the definition of a three–place ternary relation correspond to the definition of a *directed bisimulation* for categorial logic (van Benthem, 1996, Def. 12.11) (originally introduced by Kurtonina (1995)). The correspondence is not surprising since a directed bisimulation preserves the truth of the formulas of categorial logic and of the ternary modal semantics (van Benthem, 2010, Chap. 10).

Our definition of $\mathcal{L}_{\mathsf{Sub}}$–model corresponds to the definition of a *model* in (Restall, 2000, Chap. 11) stripped out from all its truth sets. These other features are not needed for what concerns us here.

Definition 10 ($\mathcal{L}_{\mathsf{Sub}}$–model)**.** A $\mathcal{L}_{\mathsf{Sub}}$*–model* is a tuple $\mathcal{M}_{\mathcal{R}} = (\mathcal{P}, \mathcal{T}, \mathcal{R}, \mathcal{I})$ where:

- $\mathcal{P} = (P, \sqsubseteq)$ is a point set;

- $\mathcal{T} \subseteq \mathcal{P} \times \mathcal{P}$ is a positive two–place accessibility relation on $\mathcal{P}$;

- $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P} \times \mathcal{P}$ is a three–place accessibility relation on $\mathcal{P}$;

- $\mathcal{I} : P \to 2^{ATM}$ is an interpretation function.

We abusively write $x \in \mathcal{M}_{\mathcal{R}}$ for $x \in \mathcal{P}$, and $(\mathcal{M}_{\mathcal{R}}, x)$ is called a *pointed* $\mathcal{L}_{\mathsf{Sub}}$*–model.* $\square$

Note that in the above definition, there could be multiple positive two–place accessibility relations $\mathcal{T}_1, \ldots, \mathcal{T}_n$ corresponding to multiple modalities $\Box_1, \ldots \Box_n$. We refrain from defining $\mathcal{L}_{\mathsf{Sub}}$–models in their full generality in order to ease the readability of the report.

Definition 11 (Truth conditions of $\mathcal{L}_{\mathsf{Sub}}$)**.** Let $\mathcal{M}_{\mathcal{R}}$ be a $\mathcal{L}_{\mathsf{Sub}}$–model, $x \in \mathcal{M}_{\mathcal{R}}$ and

$\varphi \in \mathcal{L}_{\mathsf{Sub}}$. The relation $\mathcal{M}_{\mathcal{R}}, x \Vdash \varphi$ is defined inductively as follows:

$\mathcal{M}_{\mathcal{R}}, x \Vdash \top$		always
$\mathcal{M}_{\mathcal{R}}, x \Vdash \bot$		never
$\mathcal{M}_{\mathcal{R}}, x \Vdash p$	iff	$p \in \mathcal{I}(x)$
$\mathcal{M}_{\mathcal{R}}, x \Vdash \neg\varphi$	iff	not $\mathcal{M}_{\mathcal{R}}, x \Vdash \varphi$
$\mathcal{M}_{\mathcal{R}}, x \Vdash \varphi \wedge \psi$	iff	$\mathcal{M}_{\mathcal{R}}, x \Vdash \varphi$ and $\mathcal{M}_{\mathcal{R}}, x \Vdash \psi$
$\mathcal{M}_{\mathcal{R}}, x \Vdash \varphi \vee \psi$	iff	$\mathcal{M}_{\mathcal{R}}, x \Vdash \varphi$ or $\mathcal{M}_{\mathcal{R}}, x \Vdash \psi$
$\mathcal{M}_{\mathcal{R}}, x \Vdash \Box\varphi$	iff	for all $y \in \mathcal{M}_{\mathcal{R}}$, where $x\mathcal{T}y, \mathcal{M}_{\mathcal{R}}, y \Vdash \varphi$
$\mathcal{M}_{\mathcal{R}}, x \Vdash \varphi \supset \psi$	iff	for all $y, z \in \mathcal{P}$ where $\mathcal{R}xyz$, if $\mathcal{M}_{\mathcal{R}}, y \Vdash \varphi$ then $\mathcal{M}_{\mathcal{R}}, z \Vdash \psi$
$\mathcal{M}_{\mathcal{R}}, x \Vdash \psi \subset \varphi$	iff	for all $y, z \in \mathcal{P}$ where $\mathcal{R}yxz$ if $\mathcal{M}_{\mathcal{R}}, y \Vdash \varphi$ then $\mathcal{M}_{\mathcal{R}}, z \Vdash \psi$
$\mathcal{M}_{\mathcal{R}}, x \Vdash \varphi \circ \psi$	iff	there are $y, z \in \mathcal{P}$ such that $\mathcal{R}yzx, \mathcal{M}_{\mathcal{R}}, y \Vdash \varphi$ and $\mathcal{M}_{\mathcal{R}}, z \Vdash \psi$

We extend the scope of the relation $\Vdash$ to also relate points to $\mathcal{L}_{\mathsf{Sub}}$–structures:

$\mathcal{M}_{\mathcal{R}}, x \Vdash X, Y$	iff	$\mathcal{M}_{\mathcal{R}}, x \Vdash X$ and $\mathcal{M}_{\mathcal{R}}, x \Vdash Y$
$\mathcal{M}_{\mathcal{R}}, x \Vdash X; Y$	iff	there are $y, z \in \mathcal{M}_{\mathcal{R}}$ such that $\mathcal{R}yzx, \mathcal{M}_{\mathcal{R}}, y \Vdash X$ and $\mathcal{M}_{\mathcal{R}}, z \Vdash Y$

We say that $\mathcal{M}_{\mathcal{R}}$ *validates* a $\mathcal{L}_{\mathsf{Sub}}$–structure X when for all $x \in \mathcal{M}_{\mathcal{R}}$, $\mathcal{M}_{\mathcal{R}}, x \Vdash X$. Let X be a structure and let $\varphi \in \mathcal{L}_{\mathsf{Sub}}$. We say that X *entails* φ, written $X \Vdash \varphi$, when the following holds:

$X \Vdash \varphi$ iff for all pointed $\mathcal{L}_{\mathsf{Sub}}$–model $(\mathcal{M}_{\mathcal{R}}, x)$, if $\mathcal{M}_{\mathcal{R}}, x \Vdash X$, then $\mathcal{M}_{\mathcal{R}}, x \Vdash \varphi$. $\Box$

We list below some key inferences of substructural logics, more precisely of the Lambek Calculus:

$$\varphi; \psi \Vdash \chi \quad \text{iff} \quad \varphi \Vdash \psi \supset \chi \tag{2}$$
$$\varphi \Vdash \psi \supset \chi \quad \text{iff} \quad \varphi \circ \psi \Vdash \chi \tag{3}$$
$$\varphi \circ \psi \Vdash \chi \quad \text{iff} \quad \psi \Vdash \chi \subset \varphi \tag{4}$$
$$\varphi \Vdash \psi \supset \chi \quad \text{iff} \quad \psi \Vdash \chi \subset \varphi \tag{5}$$

Urquhart's semantics. The Urquhart's semantics for relevance logic was developed independently from the Routley–Meyer's semantics in the early 1970's. An *operational frame* is a set of points $\mathcal{P}$ together with a function which gives us a new point from a pair of points:

$$\sqcup : \mathcal{P} \times \mathcal{P} \to \mathcal{P}. \tag{6}$$

An *operational model* is then an operational frame together with a relation $\Vdash$ which indicates what formulas are true at what points. The truth conditions for the implication $\supset$ are defined as follows:

$$x \Vdash \varphi \supset \psi \text{ iff for each } y\text{, if } y \Vdash \varphi \text{ then } x \sqcup y \Vdash \psi \tag{7}$$

As one can easily notice, an operational frame is a Routley-Meyer frame where $\mathcal{R}xyz$ holds if and only if $x \sqcup y = z$. Hence, the ternary relation $\mathcal{R}$ of the Routley–Meyer semantics is a generalization of the function $\sqcup$ of the Urquhart's semantics. Because it is a *relation*, it allows moreover to apply x to y and yield either a set of outcomes or no outcome at all.

3.2 Updates as Ternary Relations

The ternary relation $\mathcal{R}$ of the Routley and Meyer semantics was introduced originally for technical reasons: any 2-ary (n-ary) connective of a logical language can be given a semantics by resorting to a 3-ary (resp. $n+1$-ary) relation on worlds. Subsequently, a number of philosophical interpretations of this ternary relation have been proposed and we will briefly recall some of them at the end of this section (see (Beall et al., 2012; Restall, 2006; Mares and Meyer, 2001) for more details). However, one has to admit that providing a non-circular and conceptually grounded interpretation of this relation remains problematic. In this report, we propose a new *dynamic* interpretation of this relation, inspired by the ternary semantics of DEL.

First, one should observe that the DEL product update $\otimes$ of Definition 7 can be seen as a partial function $\mathcal{F}$ from a pair of pointed $\mathcal{L}$–model and pointed $\mathcal{L}_\alpha$–model to another pointed $\mathcal{L}$–model:

$$\mathcal{F} : \mathcal{C} \times \mathcal{C}^\alpha \to \mathcal{C} \tag{8}$$

There is a formal similarity between this abstract definition of the DEL product update and the function $\sqcup$ of Expression (6) introduced by Urquhart in the early 1970s for providing a semantics to the implication of relevance logic. This similarity is not only formal but also intuitively meaningful. Indeed, the intuitive interpretation of the DEL product update operator is very similar to the intuitive interpretation of the function $\sqcup$ of Urquhart. Points are sometimes also called "worlds", "states", "situation", "set-ups", and as explained by Restall:

> "We have a class of points (over which x and y vary), and a function $\sqcup$ which gives us new points from old. The point $x \sqcup y$ is supposed, on Urquhart's interpretation, to be the body of information given by combining x with y." (Restall, 2006, p. 363)

and also, keeping in mind the truth conditions for the connective $\supset$ of Expression (7):

> "To be committed to $A \supset B$ is to be committed to B whenever we gain the information that A. To put it another way, a body of information warrants $A \supset B$ if and only if whenever you *update* that information with new information which warrants A, the resulting (perhaps new) body of information warrants B." (my emphasis) (Restall, 2006, p. 362)

From these two quotes, it is natural to interpret the DEL product update $\otimes$ of Definition 7 as a specific kind of Urquhart's function $\sqcup$ (Expression (6)). Moreover, as explained by Restall, this substructural "update" can be nonmonotonic and may correspond to some sort of *revision*:

> "[C]ombination is sometimes nonmonotonic in a natural sense. Sometimes when a body of information is combined with another body of information, some of the original body of information might be lost. This is simplest to see in the case motivating the failure of $A \vdash B \supset A$. A body of information might tell us that A. However, when we combine it with something which tells us B, the resulting body of information might no longer warrant A (as A might with B). Combination might not simply result in the addition of information. It may well warrant its *revision*." (my emphasis) (Restall, 2006, p. 363)

Our dynamic interpretation of the ternary relation is consistent with the above considerations: sometimes, updating beliefs amounts to *revise* beliefs. As it turns out, belief revision has also been extensively studied within the DEL framework and DEL has been extended to deal with this phenomenon (Aucher, 2004; van Ditmarsch, 2005; van Benthem, 2007a; Baltag and Smets, 2008c,b; Liu, 2008; Aucher, 2008).

More generally, an update can be seen as a *partial* function $\mathcal{F}$ from a pair of pointed $\mathcal{L}$–model and pointed $\mathcal{L}_\alpha$–model to a *set* of pointed $\mathcal{L}$–model:

$$\mathcal{F} : \mathcal{C} \times \mathcal{C}^\alpha \to \mathcal{P}(\mathcal{C}) \tag{9}$$

Equivalently, an update can be seen as a ternary relation $\mathcal{R}$ defined on $\mathcal{C} \cup \mathcal{C}^\alpha$ between three pointed models $((\mathcal{M}, w), (\mathcal{E}, e), (\mathcal{M}_f, w_f))$ where $(\mathcal{M}, w)$ is a pointed $\mathcal{L}$–model, $(\mathcal{E}, e)$ is a pointed $\mathcal{L}_\alpha$–model and $(\mathcal{M}_f, w_f)$ is another pointed $\mathcal{L}$–model:

$$\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}^\alpha \times \mathcal{C} \tag{10}$$

The ternary relation of Expression (10) then resembles the ternary relation of the Routley and Meyer semantics. This is not surprising since the Routley and Meyer semantics generalizes the Urquhart semantics (they are essentially the same, since as we explained it in the previous section, an operational frame is a Routley and Meyer frame where $\mathcal{R}xyz$ holds if and only if $x \sqcup y = z$). Viewed from the perspective of DEL, the ternary relation then represents a particular sort of *update*. With this interpretation in mind, $\mathcal{R}xyz$ reads as 'the occurrence of event y in world x results in the world z' and the corresponding conditional $\alpha \supset \varphi$ reads as 'the occurrence in the current world of an event satisfying property α results in a world satisfying φ'.

The dynamic reading of the ternary relation and its corresponding conditional is very much in line with the so-called "Ramsey Test" of conditional logic. The Ramsey test can be viewed as the very first modern contribution to the logical study of conditionals and much of the contemporary work on conditional logic can be traced back to the famous footnote of Ramsey (1929). Roughly, it consists in defining a counterfactual conditional in terms of belief revision: an agent currently believes that φ would be true if ψ were true (i.e. $\psi \supset \varphi$) if and only if he should believe φ after *learning* ψ. A first attempt to provide truth conditions for conditionals, based on Ramsey's ideas, was proposed by Stalnaker. He defined his semantics by means of selection functions over possible worlds $f : W \times 2^W \to W$. As one can easily notice, Stalnaker's selection functions could also be considered from a formal point of view as a special kind of ternary relation, since a relation $\mathcal{R}_f \subseteq W \times 2^W \times W$ can be canonically associated to each selection function f. Moreover, like the ternary relation corresponding to a product update (Expression (10)), this ternary relation is 'two-sorted': the antecedent of a conditional takes value in a set of worlds (instead of a single world).[2] So, the dynamic reading of the ternary semantics is consistent with the dynamic reading of conditionals proposed by Ramsey.

This dynamic reading was not really considered and investigated by substructural logicians when they connected the substructural ternary semantics with conditional logic (Beall et al., 2012). On the other hand, the dynamic reading of inferences has been stressed to a large extent by van Benthem (2007b; 2011a) (we will come back to this point in Section 5.4), and also by Baltag and Smets (2006, 2008c,b) who distinguished *dynamic* belief revision from *static* (standard) belief revision. What distinguishes *dynamic* belief revision from *static* belief revision is that the latter is a revision of the agent's beliefs about the state of the world as it was before an event, and the former is a revision of the state of the world as it is after the event. Note, however, that this important distinction between static belief revision and dynamic belief revision collapses in the case of relevant logic, because in that case we only deal with propositional formulas. This shows again that a *dynamic* interpretation of the ternary semantics of substructural logic is consistent with the interpretations proposed by substructural logicians. In fact, our point of view is also very much in line with the claim of Gärdenfors and Makinson (1991; 1989) that non-monotonic reasoning and belief revision are "two sides of the same coin": as a matter of fact, non-monotonic reasoning is a reasoning style and belief revision is a sort of update. Likewise, the formal connection in this case also relies on a similar idea based on the Ramsey test.

To summarize our discussion, the DEL product update provides substructural logics with an intuitive and consistent interpretation of its ternary relation. This interpretation is consistent in the sense that the intuitions underlying the definitions of the DEL framework are coherent with those underlying the ternary semantics of substructural logic, as witnessed by our quotes and citations from the substructural literature.

Other interpretations of the ternary relation One interpretation, due to Barwise (1993) and developed by Restall (1996), takes worlds to be 'sites' or 'channels', a site being possibly a channel and a channel being possibly a site. If x, y and z are sites, $\mathcal{R}xyz$ reads as 'x is a channel between y and z'. Hence, if $\varphi \supset \psi$ is true at channel x, it means that all sites y and z connected by channel x are such that if φ is information available in y, then ψ is information available in z. Another similar interpretation due to Mares (1996) adapts Israel and Perry's theory of information (Perry and Israel, 1990) to the relational semantics. In this interpretation, worlds are situations in the sense of Barwise

[2]Note that Burgess (1981) already proposed a ternary semantics for conditionals, but his truth conditions and his interpretation of the ternary relation were quite different from ours.

and Perry's situation semantics (Barwise and Perry, 1983) and pieces of information – called *infons* – can carry information about other infons: an infon might carry the information that a red light on a mobile phone carries the information that the battery of the mobile phone is low. In this interpretation, the ternary relation $\mathcal{R}$ represents the informational links in situations: if there is an informational link in situation x that says that an infon σ carries the information that the infon π also holds, then if $\mathcal{R}xyz$ holds and y contains the infon σ, then z contains the infon π. Other interpretations of the ternary relation have been proposed by Beall et al. (2012), with a particular focus on their relation to conditionality.

3.3 Update logic

We define our logical language, whose semantics will be based on the idea to view an update as a ternary relation of a susbtructural frame. This idea is motivated and intuitively grounded in the analysis of the previous section.

Our language extends both the language $\mathcal{L}$ and the language $\mathcal{L}_\alpha$ of Section 2. Like our semantics, it is two-sorted: it contains both formulas of $\mathcal{L}$ and formulas of $\mathcal{L}_\alpha$.

Definition 12 (Language $\mathcal{L}_\mathcal{R}$)**.** The *language* $\mathcal{L}_\mathcal{R}$ is two-sorted and is defined by a double induction as follows:

$$\begin{array}{lllllllllllllll}
\mathcal{L}^\varphi_\mathcal{R}: & \varphi & ::= & p & \mid & \neg\varphi & \mid & \varphi\wedge\varphi & \mid & \varphi\vee\varphi & \mid & \Box_j\varphi & \mid & \alpha\supset\varphi & \mid & \varphi\circ\alpha \\
\mathcal{L}^\alpha_\mathcal{R}: & \alpha & ::= & p_\psi & \mid & \neg\alpha & \mid & \alpha\wedge\alpha & \mid & \alpha\vee\alpha & \mid & \Box_j\alpha & \mid & \varphi\subset\varphi & &
\end{array}$$

where p ranges over ATM, ψ ranges over $\mathcal{L}^\varphi_\mathcal{R}$ and j over AGT. □

Definition 13 ($\mathcal{L}_\mathcal{R}$–structure and $\mathcal{L}_\mathcal{R}$–sequent)**.** The $\mathcal{L}_\mathcal{R}$*–structures* are defined inductively as follows:

$$\begin{array}{lllllllll}
\mathcal{S}^\varphi: & X & ::= & \varphi & \mid & X,X & \mid & (X;X^\alpha) \\
\mathcal{S}^{\varphi'}: & X & ::= & \varphi & \mid & X,X & &
\end{array}$$

where φ ranges over $\mathcal{L}_\mathcal{R}$ and α ranges over $\mathcal{L}_\alpha$. The expression $\Gamma(X)$ denotes a $\mathcal{L}_\mathcal{R}$–structure containing as substructure the $\mathcal{L}_\mathcal{R}$–structure X, and $\Gamma(Z)$ denotes the $\mathcal{L}_\mathcal{R}$–structure $\Gamma(X)$ where X is uniformly substituted by the structure Z.

A $\mathcal{L}_\mathcal{R}$*–sequent* is a $\mathcal{L}_\alpha$–sequent or an expression of the form $X \models Y$ or $X \models$, where $X \in \mathcal{S}^\varphi$, $Y \in \mathcal{S}^{\varphi'}$. □

Definition 14 (Update model)**.** An *update model* is a $\mathcal{L}_{\mathsf{Sub}}$–model $\mathcal{M}_\mathcal{R} = (\mathcal{P},\mathcal{R}_1,\ldots,\mathcal{R}_m,\mathcal{R},\mathcal{I})$ where:

- $\mathcal{P} = (P,=)$ is a flat point set such that $P \subseteq \mathcal{C}\cup\mathcal{C}^\alpha$.
- $\mathcal{R}_j \subseteq \mathcal{P}\times\mathcal{P}$ is a positive two-place accessibility relation on $\mathcal{P}$ for each $j \in AGT$ such that for all $x,y \in \mathcal{P}$, where $x = (\mathcal{M}_x, w_x)$ and $y = (\mathcal{M}_y, w_y)$:

$$x \in \mathcal{R}_j(y) \text{ iff } \mathcal{M}_x = \mathcal{M}_y \text{ and } w_x \in R_j(w_y)$$

- $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P} \times \mathcal{P}$ is a ternary relation on $\mathcal{P}$ such that $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}^\alpha \times \mathcal{C}$;
- $\mathcal{I} : P \to 2^{ATM \cup ATM_\alpha}$ is an interpretation such that for all $x = (W, R_1, \ldots, R_m, I) \in \mathcal{C}$, $\mathcal{I}(x) = I(x) \in 2^{ATM}$, and for all $x = (W^\alpha, R_1^\alpha, \ldots, R_m^\alpha, I^\alpha) \in \mathcal{C}^\alpha$, $\mathcal{I}(x) = I^\alpha(x) \in 2^{ATM_\alpha}$ is a singleton.

The class of update models is denoted $\mathcal{C}_\mathcal{R}$. An *update frame* is an update model without interpretation function. $\square$

Note that the accessibility relations R_j of $\mathcal{L}$-models and $\mathcal{L}_\alpha$-models are seen in this definition as positive two-place accessibility relations $\mathcal{R}_j$. The truth conditions are the same as the ones for $\mathcal{L}_\mathcal{R}$-models:

Definition 15 (Truth conditions of $\mathcal{L}_\mathcal{R}$)**.** Let $\mathcal{M}_\mathcal{R}$ be an update model, $x \in \mathcal{M}_\mathcal{R}$ and $\varphi \in \mathcal{L}_\mathcal{R}$. The relation $\mathcal{M}_\mathcal{R}, x \Vdash \varphi$ is defined inductively as follows:

$\mathcal{M}_\mathcal{R}, x \Vdash p$	iff	$p \in \mathcal{I}(x)$
$\mathcal{M}_\mathcal{R}, x \Vdash \neg\varphi$	iff	not $\mathcal{M}_\mathcal{R}, x \Vdash \varphi$
$\mathcal{M}_\mathcal{R}, x \Vdash \varphi \wedge \psi$	iff	$\mathcal{M}_\mathcal{R}, x \Vdash \varphi$ and $\mathcal{M}_\mathcal{R}, x \Vdash \psi$
$\mathcal{M}_\mathcal{R}, x \Vdash \varphi \vee \psi$	iff	$\mathcal{M}_\mathcal{R}, x \Vdash \varphi$ or $\mathcal{M}_\mathcal{R}, x \Vdash \psi$
$\mathcal{M}_\mathcal{R}, x \Vdash \Box_j\varphi$	iff	for all $y \in \mathcal{P}$ such that $x\mathcal{R}_j y, \mathcal{M}_\mathcal{R}, y \Vdash \varphi$
$\mathcal{M}_\mathcal{R}, x \Vdash \alpha \supset \psi$	iff	for all $y, z \in \mathcal{P}$ such that $\mathcal{R}xyz$, if $\mathcal{M}_\mathcal{R}, y \Vdash \alpha$ then $\mathcal{M}_\mathcal{R}, z \Vdash \psi$
$\mathcal{M}_\mathcal{R}, x \Vdash \psi \subset \varphi$	iff	for all $y, z \in \mathcal{P}$ such that $\mathcal{R}yxz$, if $\mathcal{M}_\mathcal{R}, y \Vdash \varphi$ then $\mathcal{M}_\mathcal{R}, z \Vdash \psi$
$\mathcal{M}_\mathcal{R}, x \Vdash \varphi \circ \alpha$	iff	there are $y, z \in \mathcal{P}$ such that $\mathcal{R}yzx, \mathcal{M}_\mathcal{R}, y \Vdash \varphi$ and $\mathcal{M}_\mathcal{R}, z \Vdash \alpha$

We extend the scope of the relation $\Vdash$ to also relate points to $\mathcal{L}_\mathcal{R}$–structures:

$\mathcal{M}_\mathcal{R}, x \Vdash X, Y$	iff	$\mathcal{M}_\mathcal{R}, x \Vdash X$ and $\mathcal{M}_\mathcal{R}, x \Vdash Y$
$\mathcal{M}_\mathcal{R}, x \Vdash X; Y$	iff	there are $y, z \in \mathcal{M}_\mathcal{R}$ such that $\mathcal{R}yzx, \mathcal{M}_\mathcal{R}, y \Vdash X$ and $\mathcal{M}_\mathcal{R}, z \Vdash Y$

Let C be a class of update models, and let $X \vdash \varphi$ be a $\mathcal{L}_\mathcal{R}$–sequent. We say that X *entails* φ *in the class* C, written $X \Vdash_C \varphi$, when the following holds:

$$X \Vdash_C \varphi \quad \text{iff} \quad \text{for all } x \in \mathcal{M}_\mathcal{R} \in C, \text{ if } \mathcal{M}_\mathcal{R}, x \Vdash X, \text{ then } \mathcal{M}_\mathcal{R}, x \Vdash \varphi.$$

Let $\mathcal{M}_\mathcal{R}$ be an update model. We say that $\mathcal{M}_\mathcal{R}$ *validates a* $\mathcal{L}_\mathcal{R}$*-sequent* $X \vdash \varphi$ when $X \Vdash_{\{\mathcal{M}_\mathcal{R}\}} \varphi$. We say that $\mathcal{M}_\mathcal{R}$ *validates an inference Rule* when, if $\mathcal{M}_\mathcal{R}$ validates its premise(s), then $\mathcal{M}_\mathcal{R}$ validates its conclusion(s). Similar definitions hold by replacing models with frames. $\square$

Naturally, the truth conditions for $\Vdash$ coincide with the truth conditions for $\models$ if we only consider epistemic formulas:

Proposition 1. *Let $\mathcal{M}_\mathcal{R}$ be an update model, $\varphi \in \mathcal{L}$ and $x \in \mathcal{M}_R$ such that $x \in \mathcal{C}$. Then, $\mathcal{M}_\mathcal{R}, x \Vdash \varphi$ iff $x \models \varphi$. Let $\alpha \in \mathcal{L}_\alpha$ and let $y \in \mathcal{M}_\mathcal{R}$ such that $y \in \mathcal{C}^\alpha$. Then, $\mathcal{M}_\mathcal{R}, y \Vdash \varphi$ iff $y \models \alpha$.*

Proof. Straightforward. $\square$

Epistemic Temporal Logic. The frame semantics of substructural logic is very abstract and general and it provides a rich framework which captures a wide range of logics, such as arrow logic (van Benthem, 1996, Chap. 8), action frames and domain space (see (Restall, 2000, Example 11.12–11.15) for more details) but also linear logic, relevance logic, the lambek calculus. . . But the epistemic temporal models of ETL (Parikh and Ramanujam, 2003) (which have been related to DEL byvan Benthem et al. (2007, 2009a)) can also be viewed as models of the ternary semantics of substructural logic:

Fact 1. *Any epistemic temporal model of ETL can be mapped to an update model.*

Proof. We recall the basic definitions of ETL. Let Σ be any set. Elements of Σ are called *events*, and elements of the set of finite strings Σ^* *histories*. Given $h, h' \in \Sigma^*$, we write $h \preceq h'$ if h is a prefix of h'. A *protocol* is a prefix-closed set $\mathcal{H} \subseteq \Sigma^*$, *i.e.* $\{h \in \Sigma^* : h \text{ is non-empty and there is } h' \in \mathcal{H} \text{ such that } h \preceq h'\} \subseteq \mathcal{H}$. An *epistemic temporal model of ETL* is a tuple $(\mathcal{H}, R, V)$ where $\mathcal{H}$ is a protocol, $R : AGT \to 2^{\mathcal{H}\times\mathcal{H}}$ assigns an accessibility relation $R(j) := R_j$ to each agent $j \in AGT$, and $V : ATM \to 2^{\mathcal{H}}$ is a valuation.

An epistemic temporal model $M = (\mathcal{H}, R, V)$ can thus be naturally mapped to an update model $\mathcal{M}_{\mathcal{R}} = (\mathcal{P}, \mathcal{R}_1, \dots, \mathcal{R}_m, \mathcal{R}, \mathcal{I})$ as follows:

- $\mathcal{P} := (P, =)$ is defined by $P := P_1 \sqcup P_2$ where $P_1 := \{(M, h) : h \in \mathcal{H}\}$ and $P_2 := \{(M, h^c) : h \in \mathcal{H}^c\}$ where $\mathcal{H}^c$ is a disjoint copy of $\mathcal{H}$;
- for all $j \in AGT$, for all $h, h' \in \mathcal{H}$, $h' \in \mathcal{R}_j(h)$ iff $h' \in R_j(h)$ and for all $h^c, h^{'c} \in \mathcal{H}^c$, $h^{'c} \in \mathcal{R}_j(h^c)$ iff $h' \in R_j(h)$;
- $\mathcal{R} \subseteq P_1 \times P_2 \times P_1$ is such that $(h, h^{'c}, h'') \in \mathcal{R}$ iff $h'' = h' = he$ for some $e \in \Sigma$;
- for all $x \in P_1$, $\mathcal{I}(x) = \{p \in ATM : x \in V(p)\}$, and for all $x \in P_2$, $\mathcal{I}(x)$ is an arbitrary singleton of ATM_α. □

4 Gentzen Calculi

In this section, we provide a Gentzen calculus for update logic. The completeness proof for this Gentzen calculus relies on a new method that is not specific to our update logic but that applies to any Gentzen-like calculus. So, to highlight these new ideas, we first apply this method to the case of modal logic in Section 4.1. Then, we extend our proof to prove completeness of our Gentzen calculus for update logic in Section 4.2.

4.1 A New Method to Prove Completeness of Gentzen Calculi

We first define our Gentzen calculus for modal (epistemic) logic.

Definition 16 (Sequent Calculus L)**.** The *sequent calculus for $\mathcal{L}$*, denoted L, is defined in Figure 6. A $\mathcal{L}$–sequent $X \vdash Y$ is *provable in* L, written $X \vdash_{\mathsf{L}} Y$, when it can be derived from the axioms and inference rules of L in a finite number of steps. □

Axiom:

$\varphi \vdash \varphi$

Structural Rules:

Weakening:

$$\frac{X \vdash Y}{X, \varphi \vdash Y}\ W_L \qquad \frac{X \vdash Y}{X \vdash \varphi, Y}\ W_R$$

Contraction:

$$\frac{X, \varphi, \varphi \vdash Y}{X, \varphi \vdash Y}\ C_L \qquad \frac{X \vdash Y, \varphi, \varphi}{X \vdash Y, \varphi}\ C_R$$

Permutation:

$$\frac{X, \psi, \varphi, Y \vdash Z}{X, \varphi, \psi, Y \vdash Z}\ P_L \qquad \frac{X \vdash Y, \psi, \varphi, Z}{X \vdash Y, \varphi, \psi, Z}\ P_R$$

Cut Rule:

$$\frac{X \vdash Y, \varphi \quad \varphi, U \vdash V}{X, U \vdash Y, V}\ Cut$$

Logical Rules:

Propositional Rules:

$$\frac{X \vdash Y, \varphi}{X, \neg\varphi \vdash Y}\ \neg_R \qquad \frac{X, \varphi \vdash Y}{X \vdash Y, \neg\varphi}\ \neg_L$$

$$\frac{X, \varphi \vdash Y}{X, \varphi \wedge \psi \vdash Y}\ \wedge_L^1 \qquad \frac{X, \psi \vdash Y}{X, \varphi \wedge \psi \vdash Y}\ \wedge_L^2 \qquad \frac{X \vdash Y, \varphi \quad U \vdash V, \psi}{X, U \vdash Y, V, \varphi \wedge \psi}\ \wedge_R$$

$$\frac{X \vdash Y, \varphi}{X \vdash Y, \varphi \vee \psi}\ \vee_R^1 \qquad \frac{X \vdash Y, \psi}{X \vdash Y, \varphi \vee \psi}\ \vee_R^2 \qquad \frac{X, \varphi \vdash Y \quad U, \psi \vdash V}{X, U, \varphi \vee \psi \vdash Y, V}\ \vee_L$$

Modal Rule:

$$\frac{X \vdash \varphi}{\Box_j X \vdash \Box_j \varphi}\ k$$

where the $\mathcal{L}$–structure $\Box_j X$ is defined inductively as follows: $\Box_j X := \Box_j \varphi$ if $X = \varphi$ and $\Box_j X := \Box_j Y, \Box_j Z$ if $X = Y, Z$.

Figure 6: Sequent Calculus of Modal Logic

As one can easily notice, our sequent calculus for modal logic of Figure 7 is simply the classical sequent calculus for modal logic (without resorting to the symbol $\bot$) (Leivant, 1981; Sambin and Valentini, 1982), *i.e.*, it is the original sequent calculus LK of Gentzen (1935) (for propositional formulas and without material implication) together with the modal Rule k. Gentzen calculi for modal logic are studied in depth in (Poggiolesi, 2010; Wansing, 2002).

Theorem 2 (Soundness and completeness)**.** *For all $\mathcal{L}$-sequents $X \vdash Y$, it holds that $X \vdash_{\mathsf{L}} Y$ iff $X \Vdash Y$.*

Soundness and Completeness Proof of L w.r.t. $\mathcal{L}$ In the rest of this section, we prove Theorem 2. We use a new method compared to the usual proof method which relies on a syntactic reduction of provable sequents to theorems of a Hilbert system (Kleene et al., 1971) by resorting to the Cut Rule. The idea underlying our completeness proof is to use a Henkin style construction and define a canonical model that will invalidate the considered $\mathcal{L}$–sequent.

To prove Theorem 2, we first define the notions of *L–consistent set* over the set $\mathcal{L}$. First, we introduce some notations. If $Y = \varphi_1, \ldots, \varphi_n \in \mathcal{S}$, then $\neg Y$ denotes $\neg\varphi_1, \ldots, \neg\varphi_n$.

Definition 17 (L–consistent set and maximal L–consistent set of $\mathcal{L}$)**.**

- A *L–consistent set* Γ *of* $\mathcal{L}$ is a set of formulas of $\mathcal{L}$ such that there are no $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{L}}$.
- A *maximal L–consistent set* Γ *of* $\mathcal{L}$ is a L–consistent set of $\mathcal{L}$ such that there is no $\varphi \in \mathcal{L}$ such that $\varphi \notin \Gamma$ and $\Gamma \cup \{\varphi\}$ is L–consistent. □

Fact 2. *Let Γ be a maximal L–consistent set of $\mathcal{L}$. Then, for all $\varphi \in \mathcal{L}$, $\varphi \vee \neg\varphi \in \Gamma$.*

Proof. Assume that there is $\varphi \in \mathcal{L}$ such that $\varphi \vee \neg\varphi \notin \Gamma$. Then, $\Gamma \cup \{\varphi \vee \neg\varphi\}$ is not L-consistent. So, there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\varphi \vee \neg\varphi, \varphi_1, \ldots, \varphi_n \vdash_{\mathsf{L}}$. However, $\vdash_{\mathsf{L}} \neg\varphi \vee \varphi$. So, by application of the Cut rule, we have that $\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{L}}$. But because $\varphi_1, \ldots, \varphi_n \in \Gamma$, this entails that Γ is not L–consistent, which is impossible. □

Lemma 3. *Let Γ be a maximal L–consistent set. For all $\varphi_1, \ldots, \varphi_k \in \Gamma$, all $\varphi \in \mathcal{L}$, if $\varphi_1, \ldots, \varphi_k \vdash_{\mathsf{L}} \varphi$ then $\varphi \in \Gamma$.*

Proof. First, we show that $\Gamma \cup \{\varphi\}$ is L–consistent. Assume towards a contradiction that it is not the case. Then, there are $\psi_1, \ldots, \psi_l \in \Gamma$ such that $\psi_1, \ldots, \psi_l, \varphi \vdash_{\mathsf{L}}$. By W_L and W_R, we have that $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_l, \varphi \vdash_{\mathsf{L}}$ and $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_l \vdash_{\mathsf{L}} \varphi$, because by assumption $\varphi_1, \ldots, \varphi_k \vdash_{\mathsf{L}} \varphi$. Therefore, $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_l, \neg\varphi \vdash_{\mathsf{L}}$ by Rule $\neg_L$. So, $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_l, \neg\varphi \vee \varphi \vdash_{\mathsf{L}}$ by Rule $\vee_L$. Because $\neg\varphi \vee \varphi \in \Gamma$ by Fact 2 (since Γ is a maximal L–consistent set), and $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_l \in \Gamma$, we conclude that Γ is not L–consistent, which is impossible. Thus, our initial assumption was wrong and $\Gamma \cup \{\varphi\}$ is L–consistent. Then, because Γ is a *maximal* L–consistent set, we have finally that $\varphi \in \Gamma$. □

Then, we have the following Lindenbaum-like lemma:

Lemma 4. *Any* L*–consistent set over* $\mathcal{L}$ *can be extended into a* maximal L*–consistent set over* $\mathcal{L}$*.*

Proof. Let Γ be a L–consistent set of $\mathcal{L}$ and let $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ be an enumeration of the formulas of $\mathcal{L}$ (it exists because ATM is countable). We define inductively the sets Γ_n as follows:

$$\begin{aligned} \Gamma_0 &:= \Gamma \\ \Gamma_{n+1} &:= \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is } \mathsf{L}\text{–consistent;} \\ \Gamma_n & \text{otherwise} \end{cases} \end{aligned}$$

Then, we define the set Γ^+ of $\mathcal{L}$ as follows: $\Gamma^+ := \bigcup\limits_{n \geq 0} \Gamma_n$.

We show that Γ^+ is a maximal L–consistent set of $\mathcal{L}$. Clearly, for all $n \in \mathbb{N}$, Γ_n is L-consistent by definition of Γ_n. So, if Γ^+ was not L-consistent, there would be $n_0 \in \mathbb{N}$ such that Γ_{n_0} is not L-consistent, which is impossible. Now, assume towards a contradiction that Γ^+ is not a maximal L-consistent set. Then, there is $\varphi \in \mathcal{L}$ such that $\varphi \notin \Gamma^+$ and $\Gamma^+ \cup \{\varphi\}$ is L-consistent. But there is $n \in \mathbb{N}$ such that $\varphi = \varphi_n$. Because $\varphi \notin \Gamma^+$, we also have that $\varphi_n \notin \Gamma_{n+1}$. So, $\Gamma_n \cup \{\varphi_n\}$ is not L-consistent by definition of Γ^+. Therefore, $\Gamma^+ \cup \{\varphi\}$ is not L-consistent, which is impossible. $\square$

Then, we define the canonical update model associated to L.

Definition 18 (Canonical update model)**.** The *canonical update model associated to* L is the L-model $\mathcal{M}_c := (S^c, R_1^c, \ldots R_m^c, I^c)$ defined as follows:

- S^c is the set $\mathcal{C}$ of all maximal L–consistent sets of $\mathcal{L}$;
- for all $\Gamma, \Gamma' \in \mathcal{C}$, all $j = 1, \ldots, m$,

$$(\Gamma, \Gamma') \in R_j^c \quad \text{iff} \quad \text{for all } \Box_j \varphi \in \Gamma, \text{ we have that } \varphi \in \Gamma'$$

- for all $p \in ATM$, $p \in \mathcal{I}_c(\Gamma)$ iff $p \in \Gamma$. $\square$

Lemma 5. *It holds that* $\mathcal{R}_{\supset} = \mathcal{R}_{\subset} = \mathcal{R}_{\circ}$*.*

Proof. The proof is similar to the proof of Lemma 11.25 of (Restall, 2000). $\square$

Hence, from now on, we will denote this relation $\mathcal{R} := \mathcal{R}_{\supset} = \mathcal{R}_{\subset} = \mathcal{R}_{\circ}$.

Lemma 6 (Truth lemma)**.** *For all* $\varphi \in \mathcal{L}$*, for all maximal consistent set* Γ*, we have that*

$$\mathcal{M}^c, \Gamma \Vdash \varphi \quad \textit{iff} \quad \varphi \in \Gamma. \tag{11}$$

Proof. The proof is by induction on φ.

- $\varphi := p$: this case holds by definition of $\mathcal{M}^c$.

- $\varphi := \neg\psi$: assume that $\neg\psi \in \Gamma$. Assume towards a contradiction that $\psi \in \Gamma$. By application of Rule $\neg_R$ to the Axiom, we have that $\psi, \neg\psi \models_{\mathsf{L}}$. But $\psi, \neg\psi \in \Gamma$, so Γ is not L-consistent. This is impossible, so $\psi \notin \Gamma$. Therefore, by induction hypothesis, it is not the case that $\Gamma \models \psi$. That is, $\Gamma \models \neg\psi$. For the other direction, if $\Gamma \models \neg\psi$, then it is not the case that $\Gamma \models \psi$. So, by induction hypothesis, $\psi \notin \Gamma$. If $\neg\psi \notin \Gamma$, then $\psi \notin \Gamma$ and $\neg\psi \notin \Gamma$. Then, $\Gamma \cup \{\psi\}$ and $\Gamma \cup \{\neg\psi\}$ are not L–consistent because Γ is a maximal consistent set of $\mathcal{L}$. So, there are $\varphi_1, \dots, \varphi_n \in \Gamma$, $\psi_1, \dots, \psi_m \in \Gamma$ such that $\varphi_1, \dots, \varphi_n, \psi \models_{\mathsf{L}}$ and $\psi_1, \dots, \psi_m, \neg\psi \models_{\mathsf{L}}$. Therefore, by Rule W_L, $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m, \varphi \models_{\mathsf{L}}$ and $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m, \neg\varphi \models_{\mathsf{L}}$ .So, by Rule $\vee_L$, $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m, \varphi \vee \neg\varphi \models_{\mathsf{L}}$. However, $\neg\psi \vee \psi \in \Gamma$ by Fact 2 and $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m \in \Gamma$, so Γ is not L–consistent, which is impossible. So, $\neg\psi \in \Gamma$, which proves the other direction.

- $\varphi := \varphi_1 \wedge \varphi_2$: $\mathcal{M}^c, \Gamma \models \varphi_1 \wedge \varphi_2$ iff $\mathcal{M}^c, \Gamma \models \varphi_1$ and $\mathcal{M}^c, \Gamma \models \Gamma_2$ by definition, iff $\varphi_1 \in \Gamma$ and $\varphi_2 \in \Gamma$ by induction hypothesis. Now, we prove that $\varphi_1 \in \Gamma$ and $\varphi_2 \in \Gamma$ iff $\varphi_1 \wedge \varphi_2 \in \Gamma$.

  First, we prove that if $\varphi_1 \in \Gamma$ and $\varphi_2 \in \Gamma$ then $\varphi_1 \wedge \varphi_2 \in \Gamma$. By the Axiom, we have that $\varphi_1 \models_{\mathsf{L}} \varphi_1$ and $\varphi_2 \models_{\mathsf{L}} \varphi_2$. So, by Rule W_L, we have that $\varphi_1, \varphi_2 \models_{\mathsf{L}} \varphi_1$ and $\varphi_1, \varphi_2 \models_{\mathsf{L}} \varphi_2$. Then, by $\wedge_R$, $\varphi_1, \varphi_2 \models_{\mathsf{L}} \varphi_1 \wedge \varphi_2$. But $\varphi_1, \varphi_2 \in \Gamma$. Therefore, by Lemma 3, we have that $\varphi_1 \wedge \varphi_2 \in \Gamma$.

  Second, we prove that $\varphi_1 \wedge \varphi_2 \in \Gamma$ entails that $\varphi_1 \in \Gamma$ and $\varphi_2 \in \Gamma$. Assume that $\varphi_1 \wedge \varphi_2 \in \Gamma$ and assume towards a contradiction that $\varphi_1 \notin \Gamma$ (the proof when $\varphi_2 \notin \Gamma$ is similar). Then, because Γ is a maximal consistent set, $\Gamma \cup \{\varphi_1\}$ is not L–consistent. Then, there are $\psi_1, \dots, \psi_n \in \Gamma$ such that $\varphi_1, \psi_1, \dots, \psi_n \models_{\mathsf{L}}$. Therefore, $\varphi_1 \wedge \varphi_2, \psi_1, \dots, \psi_n \models_{\mathsf{L}}$ by $\wedge_L^1$. But $\psi_1, \dots, \psi_k \in \Gamma$ and $\varphi_1 \wedge \varphi_2 \in \Gamma$. So, Γ is not L–consistent, which is impossible by assumption. Therefore, $\varphi_1 \in \Gamma$. We prove similarly that $\varphi_2 \in \Gamma$.

- $\varphi := \varphi_1 \vee \varphi_2$: $\mathcal{M}^c, \Gamma \models \varphi_1 \vee \varphi_2$ iff $\mathcal{M}^c, \Gamma \models \varphi_1$ or $\mathcal{M}^c, \Gamma \models \varphi_2$ by definition, iff $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$ by induction hypothesis. Now, we prove that $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$ iff $\varphi_1 \vee \varphi_2 \in \Gamma$.

  First, we prove that if $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$ then $\varphi_1 \vee \varphi_2 \in \Gamma$. Assume towards a contradiction that $\varphi_1 \vee \varphi_2 \notin \Gamma$. Then, $\Gamma \cup \{\varphi_1 \vee \varphi_2\}$ is not L–consistent. So, there are $\psi_1, \dots, \psi_n \in \Gamma$ such that $\varphi_1 \vee \varphi_2, \psi_1, \dots, \psi_n \models_{\mathsf{L}}$. However, $\varphi_1 \models_{\mathsf{L}} \varphi_1 \vee \varphi_2$. So, by the Cut rule, $\varphi_1, \psi_1, \dots, \psi_n \models_{\mathsf{L}}$. Likewise, because $\varphi_2 \models_{\mathsf{L}} \varphi_1 \vee \varphi_2$, we have by the Cut rule that $\varphi_2, \psi_1, \dots, \psi_n \models_{\mathsf{L}}$. Therefore, in both cases ($\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$), we obtain that Γ is not L–consistent, because $\psi_1, \dots, \psi_n \in \Gamma$. This is impossible. Thus, $\varphi_1 \vee \varphi_2 \in \Gamma$.

  Second, we prove that if $\varphi_1 \vee \varphi_2 \in \Gamma$ then $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$. Assume that $\varphi_1 \notin \Gamma$ and that $\varphi_2 \notin \Gamma$. Then, $\Gamma \cup \{\varphi_1\}$ is not L–consistent because Γ is a maximal consistent set. So, there are $\psi_1, \dots, \psi_n \in \Gamma$ such that $\varphi_1, \psi_1, \dots, \psi_n \models_{\mathsf{L}}$. Likewise, $\Gamma \cup \{\varphi_2\}$ is not L–consistent because Γ is a maximal L–consistent set. So, there are $\psi'_1, \dots, \psi'_m \in \Gamma$ such that $\varphi_2, \psi'_1, \dots, \psi'_l \models_{\mathsf{L}}$.

Then, $\varphi_1, \psi_1, \ldots, \psi_n, \psi'_1, \ldots, \psi'_m \mid\!\frac{}{\mathsf{L}}$ by Rules W_L and $\varphi_1, \psi_1, \ldots, \psi_k, \psi'_1, \ldots, \psi'_l \mid\!\frac{}{\mathsf{L}}$ also by W_L. Hence, $\varphi_1 \vee \varphi_2, \psi_1, \ldots, \psi_k, \psi'_1, \ldots, \psi'_l \mid\!\frac{}{\mathsf{L}}$ by $\vee_L$. However, $\varphi_1 \vee \varphi_2, \psi_1, \ldots, \psi_k, \psi'_1, \ldots, \psi'_l \in \Gamma$. So, Γ is not L–consistent. This is impossible. So, either $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$.

- $\varphi := \Box_j \psi$: assume that $\mathcal{M}_c, \Gamma \models \Box_j \psi$. We are going to show that $\Box_j \psi \in \Gamma$. Let $S := \{\neg\psi\} \cup \{\varphi : \Box_j \varphi \in \Gamma\}$ and assume that S satisfies (1). Then, S can be extended to a maximal L–consistent set S^+ by Lemma 4. Now, $\psi \notin S^+$ because $\neg\psi \in S$. So, $\mathcal{M}_c, S^+ \models \neg\psi$. Moreover, $S^+ \in R_j(\Gamma)$ by definition of R_j. Hence, $\mathcal{M}_c, \Gamma \models \Diamond_j \neg\psi$. This is impossible by assumption. So, S is not L–consistent. So, there are $\varphi_1, \ldots, \varphi_n \in S$ such that $\neg\psi, \varphi_1, \ldots, \varphi_n \mid\!\frac{}{\mathsf{L}}$. Then, $\varphi_1, \ldots, \varphi_n \mid\!\frac{}{\mathsf{L}} \neg\neg\psi$ by Rule $\neg_L$. So, by Rule k, $\Box_j \varphi_1, \ldots, \Box_j \varphi_n \mid\!\frac{}{\mathsf{L}} \Box_j \neg\neg\psi$. Moreover, $\Box_j \varphi_1, \ldots, \Box_j \varphi_n \in \Gamma$. So, because Γ is a maximal consistent set of $\mathcal{L}$, $\Box_j \neg\neg\psi \in \Gamma$ by Lemma 3. Moreover, $\neg\neg\psi \mid\!\frac{}{\mathsf{L}} \psi$, so $\Box_j \neg\neg\psi \mid\!\frac{}{\mathsf{L}} \Box_j \psi$ by Rule k. So, again by application of Lemma 3, $\Box_j \psi \in \Gamma$.

  Assume that $\Box_j \psi \in \Gamma$. Then, for all Γ' such that $\Gamma' \in R_j(\Gamma)$, we have that $\psi \in \Gamma'$. Therefore, $\mathcal{M}_c, \Gamma' \models \psi$. So, $\mathcal{M}_c, \Gamma \models \Box_j \psi$ by definition. □

Proof of Theorem 2. The proof of soundness is routine, so we only prove completeness, *i.e.*, we prove that for all $\mathcal{L}_{\mathcal{R}}$–sequent $X \vdash Y$, if $X \Vdash Y$ holds then $X \mid\!\frac{}{\mathsf{L}} Y$ holds. Assume towards a contradiction that $X \Vdash Y$ and that it is not the case that $X \mid\!\frac{}{\mathsf{L}} Y$. However, $X, \neg Y \mid\!\frac{}{\mathsf{L}}$ iff $X \mid\!\frac{}{\mathsf{L}} Y$ by application of Rule $\neg_L$, the Cut rule and using the fact that $\neg\neg Y \mid\!\frac{}{\mathsf{L}} Y$. So, it is not the case that the set of formulas $\{X, \neg Y\}$ is L-consistent. So, by Lemma 4, it can be extended into a maximal L-consistent set Γ. Then, by the Truth Lemma 6, we have that $\mathcal{M}_c, \Gamma \models X, \neg Y$. Hence, it is not the case that $X \Vdash Y$. This contradicts our assumption and this completes the proof. □

4.2 A Gentzen Calculus for Update Logic

In this section, we introduce a sound and complete Gentzen calculus for update logic.

Definition 19 (Sequent calculus $\mathsf{L}_{\mathcal{R}}$)**.** The *sequent calculus for* $\mathcal{L}_{\mathcal{R}}$, denoted $\mathsf{L}_{\mathcal{R}}$, is defined by adding to the base sequent calculus for modal logic of Figure 7 the rules for the substructural connectives given in Figure 8. Note that the sequents of $\mathsf{L}_{\mathcal{R}}$ are all $\mathcal{L}_{\mathcal{R}}$–sequents, except for the *Modal Rule* where they are either $\mathcal{L}$–sequents or $\mathcal{L}_\alpha$–sequents. A $\mathcal{L}_{\mathcal{R}}$–sequent $X \vdash Y$ is *provable in* $\mathsf{L}_{\mathcal{R}}$, written $X \mid\!\frac{}{\mathsf{L}_{\mathcal{R}}} Y$, when it can be derived from the axioms and inference rules of $\mathsf{L}_{\mathcal{R}}$ in a finite number of steps. □

Theorem 7 (Soundness and completeness)**.** *For all* $\mathcal{L}_{\mathcal{R}}$*-sequents* $X \vdash Y$*, it holds that* $X \mid\!\frac{}{\mathsf{L}_{\mathcal{R}}} Y$ *iff* $X \Vdash Y$*.*

In the rest of this section, we prove Theorem 7. First, we define the notions of $\mathsf{L}_{\mathcal{R}}$–*consistent set* and *maximal* $\mathsf{L}_{\mathcal{R}}$*–consistent set* over the sets of $\mathcal{L}_{\mathcal{R}}$–structures $\mathcal{S}^\varphi$ and the sets of $\mathcal{L}_\alpha$–structures $\mathcal{S}^\alpha$. To do so, we first introduce the following notation, which is very similar to the notation used in the previous section for the modal language $\mathcal{L}$: if $Y = \varphi_1, \ldots, \varphi_n \in \mathcal{S}^\alpha \cup \mathcal{S}^{\varphi'}$, then $\neg Y$ denotes $\neg\varphi_1, \ldots, \neg\varphi_n$.

Axioms:

$$\varphi \vdash \varphi \qquad\qquad p_\varphi \vdash \neg p_\psi \quad \text{if } \psi \neq \varphi$$

Structural Rules:

Weakening: *Contraction:* *Permutation:*

$$\frac{\Gamma(X) \vdash Y}{\Gamma(Z, X) \vdash Y}\ W_L \qquad \frac{\Gamma(X, X) \vdash Y}{\Gamma(X) \vdash Y}\ C_L \qquad \frac{\Gamma(Y, X) \vdash Z}{\Gamma(X, Y) \vdash Z}\ P_L$$

$$\frac{X \vdash Y}{X \vdash Y, Z}\ W_R \qquad \frac{X \vdash Y, Z, Z}{X \vdash Y, Z}\ C_R \qquad \frac{X \vdash Y, \psi, \varphi, Z}{X \vdash Y, \varphi, \psi, Z}\ P_R$$

Cut Rule:

$$\frac{X \vdash \varphi \quad \Gamma(\varphi) \vdash Y}{\Gamma(X) \vdash Y}\ Cut$$

Logical Rules:

Propositional Rules:

$$\frac{X \vdash Y, \varphi}{X, \neg\varphi \vdash Y}\ \neg_R \qquad \frac{X, \varphi \vdash Y}{X \vdash Y, \neg\varphi}\ \neg_L$$

$$\frac{\Gamma(\varphi) \vdash Y}{\Gamma(\varphi \wedge \psi) \vdash Y}\ \wedge_L^1 \qquad \frac{\Gamma(\psi) \vdash Y}{\Gamma(\varphi \wedge \psi) \vdash Y}\ \wedge_L^2 \qquad \frac{X \vdash Y, \varphi \quad X \vdash Y, \psi}{X \vdash Y, \varphi \wedge \psi}\ \wedge_R$$

$$\frac{X \vdash Y, \varphi}{X \vdash Y, \varphi \vee \psi}\ \vee_R^1 \qquad \frac{X \vdash Y, \psi}{X \vdash Y, \varphi \vee \psi}\ \vee_R^2 \qquad \frac{\Gamma(\varphi) \vdash Y \quad \Gamma(\psi) \vdash Y}{\Gamma(\varphi \vee \psi) \vdash Y}\ \vee_L$$

Modal Rule:

$$\frac{X \vdash \varphi}{\Box_j X \vdash \Box_j \varphi}\ k$$

where the $\mathcal{L}$–structure $\Box_j X$ is defined inductively as follows: $\Box_j X := \Box_j \varphi$ if $X = \varphi$ and $\Box_j X := \Box_j Y, \Box_j Z$ if $X = Y, Z$.

Figure 7: Sequent Calculus $\mathsf{L}_{\mathcal{R}}$: Axioms, Structural Rules and Rules for $\neg, \wedge, \vee, \Box_j$

$$\frac{X;\alpha \vdash \varphi}{X \vdash \alpha \supset \varphi} \supset_R \qquad \frac{X^\alpha \vdash \alpha \quad \Gamma(\varphi) \vdash X}{\Gamma(\alpha \supset \varphi; X^\alpha) \vdash X} \supset_L$$

$$\frac{X \vdash \varphi \quad X^\alpha \vdash \alpha}{X; X^\alpha \vdash \varphi \circ \alpha} \circ_R \qquad \frac{\Gamma(\varphi;\alpha) \vdash \psi}{\Gamma(\varphi \circ \alpha) \vdash \psi} \circ_L$$

$$\frac{\varphi; X^\alpha \vdash \psi}{X^\alpha \vdash \psi \subset \varphi} \subset_R \qquad \frac{X \vdash \varphi \quad \Gamma(\psi) \vdash Y}{\Gamma(X; \psi \subset \varphi) \vdash Y} \subset_L$$

Figure 8: Sequent Calculus $\mathsf{L}_\mathcal{R}$: Rules for $\circ, \supset, \subset$

Definition 20 ($\mathsf{L}_\mathcal{R}$–consistent set and maximal $\mathsf{L}_\mathcal{R}$–consistent set of $\mathcal{S}^\varphi$ and $\mathcal{S}^\alpha$)**.**

- A *$\mathsf{L}_\mathcal{R}$–consistent set* Γ *of* $\mathcal{S}^\varphi$ is a set of $\mathcal{L}_\mathcal{R}$–structures of $\mathcal{S}^\varphi$ such that there is no $X \in \mathcal{S}^\varphi$ and $Y \in \mathcal{S}^{\varphi'}$ such that $X \models_{\mathsf{L}_\mathcal{R}} \neg Y$.
- A *maximal $\mathsf{L}_\mathcal{R}$–consistent set of* $\mathcal{S}^\varphi$ is a consistent set Γ of $\mathcal{S}^\varphi$ such that there is no $X \in \mathcal{S}^\varphi$ such that $X \notin \Gamma$ and $\Gamma \cup \{X\}$ is $\mathsf{L}_\mathcal{R}$–consistent.

Similar definitions hold for the $\mathcal{L}_\alpha$–structures $\mathcal{S}^\alpha$. □

Some preliminary results. We prove some facts and lemmata which will be used in the very completeness proof.

Lemma 8. *Let Γ be a maximal $\mathsf{L}_\mathcal{R}$–consistent set. For all $\varphi_1, \ldots, \varphi_k \in \Gamma$, all $\varphi \in \mathcal{L}_\mathcal{R}$, if $\varphi_1, \ldots, \varphi_k \models_{\mathsf{L}_\mathcal{R}} \varphi$ then $\varphi \in \Gamma$.*

Proof. The proof follows exactly the same reasoning as Lemma 3. □

Now, we need to prove new results in order to deal with the new connectives $\circ, \supset, \subset$ and ";" of update logic.

Definition 21. We define inductively the following translation t from $\mathcal{L}_\mathcal{R}$–structures and $\mathcal{L}_\alpha$–structures to formulas of $\mathcal{L}_\mathcal{R}$:

$$\begin{array}{lcl} t(\varphi) & = & \varphi \\ t(X,Y) & = & t(X) \wedge t(Y) \\ t(X;Y) & = & t(X) \circ t(Y) \quad \square \end{array}$$

Fact 3. *For all update models $\mathcal{M}_\mathcal{R}$, all $x \in \mathcal{M}_\mathcal{R}$ and all $\mathcal{L}_\mathcal{R}$–structure or $\mathcal{L}_\alpha$–structure X,*

$$\mathcal{M}_\mathcal{R}, x \Vdash X \quad \textit{iff} \quad \mathcal{M}_\mathcal{R}, x \Vdash t(X). \tag{12}$$

Let Γ be a maximal $\mathsf{L}_\mathcal{R}$-consistent set of $\mathcal{L}_\mathcal{R}$–structures or $\mathcal{L}_\alpha$–structures. Then, for all $\mathcal{L}_\mathcal{R}$–structures or $\mathcal{L}_\alpha$–structures X, it holds that

$$t(X) \in \Gamma \quad \textit{iff} \quad X \in \Gamma. \tag{13}$$

Proof. The proof of Expression (12) is by a direct induction on X. We prove the direction from left to right of Expression (13). Assume that $t(X) \in \Gamma$ and that $\Gamma \cup \{X\}$ is not $\mathsf{L}_{\mathcal{R}}$-consistent. Then, there is $Y \in \mathcal{S}^{\varphi}$ such that $Y, X \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}}$. So, $Y, t(X) \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}}$ by rules $\circ_L$, $\wedge^1_L$, $\wedge^2_L$ and C_L. But then, because $t(X) \in \Gamma$, Γ is not $\mathsf{L}_{\mathcal{R}}$-consistent, which is impossible. So, $\Gamma \cup \{X\}$ is $\mathsf{L}_{\mathcal{R}}$-consistent. Therefore, because Γ is a *maximal* $\mathsf{L}_{\mathcal{R}}$-consistent set, we have that $X \in \Gamma$.

Now, we prove the direction from right to left of Expression (13) by contraposition. Assume that $t(X) \notin \Gamma$. Then, because Γ is a maximal $\mathsf{L}_{\mathcal{R}}$–consistent set, there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $t(X), \varphi_1, \ldots, \varphi_n \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}}$. So, $\varphi_1, \ldots, \varphi_n \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}} \neg t(X)$ by $\neg_R$, and therefore $\neg t(X) \in \Gamma$ by Lemma 8. However, $X \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}} t(X)$, so $X, \neg t(X) \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}}$ by $\neg_L$, and $X, \neg t(X) \in \Gamma \cup \{X\}$. Therefore, $\Gamma \cup \{X\}$ is not $\mathsf{L}_{\mathcal{R}}$-consistent. Hence, $X \notin \Gamma$. This proves the second direction. □

Fact 4. *The following axioms and inference rules are derivable in $\mathsf{L}_{\mathcal{R}}$:*

$$\frac{\Gamma((X; X^\alpha), (X; Y^\alpha)) \vdash Y}{\Gamma(X; (X^\alpha, Y^\alpha)) \vdash Y}\ (;/,) \tag{14}$$

$$\frac{\Gamma((X; X^\alpha), (Y; X^\alpha)) \vdash Z}{\Gamma((X, Y); X^\alpha) \vdash Z}\ (,/;) \tag{15}$$

$$\varphi \circ (\alpha \vee \beta) \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}} (\varphi \circ \alpha) \vee (\varphi \circ \beta) \tag{16}$$

$$(\varphi \vee \psi) \circ \alpha \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}} (\varphi \circ \alpha) \vee (\psi \circ \alpha) \tag{17}$$

$$\alpha_1 \supset \varphi, \ldots, \alpha_n \supset \varphi \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}} (\alpha_1 \vee \ldots \vee \alpha_n) \supset \varphi \tag{18}$$

$$\psi \subset \varphi_1, \ldots, \psi \subset \varphi_n \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}} \psi \subset (\varphi_1 \vee \ldots \vee \varphi_n) \tag{19}$$

Proof. We prove the derivability of each axiom or inference rule:

- The proofs of Rule $(;/,)$ of Expression (14) and of Rule $(,/;)$ of Expression (15) are by application of rules W_L and C_L.
- Now, we give the full proof of $\varphi \circ (\alpha \vee \beta) \mid\!\frac{\quad}{\mathsf{L}_{\mathcal{R}}} (\varphi \circ \alpha) \vee (\varphi \circ \beta)$: the proof of Expression (17) is similar.

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi \quad \alpha \vdash \alpha}{\varphi; \alpha \vdash \varphi \circ \alpha}\ \circ_R}{\varphi; \alpha \vdash (\varphi \circ \alpha) \vee (\varphi \circ \beta)}\ W_R, \vee_R \qquad \dfrac{\dfrac{\varphi \vdash \varphi \quad \beta \vdash \beta}{\varphi; \beta \vdash \varphi \circ \beta}\ \circ_R}{\varphi; \alpha \vdash (\varphi \circ \alpha) \vee (\varphi \circ \beta)}\ W_R, \vee_R}{\varphi; \alpha \vee \beta \vdash (\varphi \circ \alpha) \vee (\varphi \circ \beta)}\ \vee_L}{\varphi \circ (\alpha \vee \beta) \vdash (\varphi \circ \alpha) \vee (\varphi \circ \beta)}\ \circ_L$$

- By iterated application of $\supset_L$ to $\alpha_1 \vdash_{\mathsf{L}_\mathcal{R}} \alpha_1, \ldots, \alpha_n \vdash_{\mathsf{L}_\mathcal{R}} \alpha_n$ and $\underbrace{\varphi, \ldots, \varphi}_{n\ times} \vdash_{\mathsf{L}_\mathcal{R}} \varphi$, we obtain that $(\alpha_1 \supset \varphi; \alpha_1), \ldots, (\alpha_n \supset \varphi; \alpha_n) \vdash_{\mathsf{L}_\mathcal{R}} \varphi$. Therefore, $(\alpha_1 \supset \varphi, \ldots, \alpha_n \supset \varphi; \alpha_1), \ldots, (\alpha_n \supset \varphi, \ldots, \alpha_n \supset \varphi; \alpha_n) \vdash_{\mathsf{L}_\mathcal{R}} \varphi$ by Rule W_L. So, $\alpha_1 \supset \varphi, \ldots, \alpha_n \supset \varphi; \alpha_1 \vee \ldots \vee \alpha_n \vdash_{\mathsf{L}_\mathcal{R}} \varphi$ by Rule $\vee_L$. So, finally, $\alpha_1 \supset \varphi, \ldots, \alpha_n \supset \varphi \vdash_{\mathsf{L}_\mathcal{R}} \alpha_1 \vee \ldots \vee \alpha_n \supset \varphi$ by Rule $\supset_R$.

- Finally, we prove that $\psi \subset \varphi_1, \ldots, \psi \subset \varphi_n \vdash_{\mathsf{L}_\mathcal{R}} \psi \subset (\varphi_1 \vee \ldots \vee \varphi_n)$. By iterated application of $\subset_L$ to $\varphi_1 \vdash_{\mathsf{L}_\mathcal{R}} \varphi_1, \ldots, \varphi_n \vdash_{\mathsf{L}_\mathcal{R}} \varphi_n$ and $\underbrace{\psi, \ldots, \psi}_{n\ times} \vdash_{\mathsf{L}_\mathcal{R}} \psi$, we have that $(\varphi_1; \psi \subset \varphi_1), \ldots, (\varphi_n; \psi \subset \varphi_n) \vdash_{\mathsf{L}_\mathcal{R}} \psi$. Therefore, by Rule W_L, we have that $(\varphi_1; \psi \subset \varphi_1, \ldots, \psi \subset \varphi_n), \ldots, (\varphi_n; \psi \subset \varphi_1, \ldots, \psi \subset \varphi_n) \vdash_{\mathsf{L}_\mathcal{R}} \psi$. So, by Rule $\vee_L$, $(\varphi_1 \vee \ldots \vee \varphi_n; \psi \subset \varphi_1, \ldots, \psi \subset \varphi_n), \ldots, (\varphi_1 \vee \ldots \vee \varphi_n; \psi \subset \varphi_1, \ldots, \psi \subset \varphi_n) \vdash_{\mathsf{L}_\mathcal{R}} \psi$. So, by C_L, $\varphi_1 \vee \ldots \vee \varphi_n; \psi \subset \varphi_1, \ldots, \psi \subset \varphi_n) \vdash_{\mathsf{L}_\mathcal{R}} \psi$. Finally, by $\subset_R$, we obtain that $\psi \subset \varphi_1, \ldots, \psi \subset \varphi_n \vdash_{\mathsf{L}_\mathcal{R}} \psi \subset (\varphi_1 \vee \ldots \vee \varphi_n)$. □

Fact 5. *Let $\varphi \in \mathcal{L}^\varphi_\mathcal{R}$ and $\alpha \in \mathcal{L}^\alpha_\mathcal{R}$. If $\varphi \circ \alpha$ is $\mathsf{L}_\mathcal{R}$-consistent, then φ is $\mathsf{L}_\mathcal{R}$-consistent and α is $\mathsf{L}_\mathcal{R}$-consistent.*

Proof. We prove that φ is $\mathsf{L}_\mathcal{R}$-consistent and then we do it for α:

- Assume towards a contradiction that φ is not $\mathsf{L}_\mathcal{R}$-consistent, that is $\varphi \vdash_{\mathsf{L}_\mathcal{R}}$. Then, we have the following reasoning:

$$\dfrac{\dfrac{\dfrac{\varphi \vdash_{\mathsf{L}_\mathcal{R}}}{\varphi \vdash_{\mathsf{L}_\mathcal{R}} \alpha \supset \varphi}\ C_R \qquad \dfrac{\alpha \vdash_{\mathsf{L}_\mathcal{R}} \alpha \quad \varphi \vdash_{\mathsf{L}_\mathcal{R}}}{\alpha \supset \varphi; \alpha \vdash_{\mathsf{L}_\mathcal{R}}}\ \supset_L}{\varphi; \alpha \vdash_{\mathsf{L}_\mathcal{R}}}\ Cut}{\varphi \circ \alpha \vdash_{\mathsf{L}_\mathcal{R}}}\ \circ_L$$

  As a consequence, we obtain that $\varphi \circ \alpha$ is not $\mathsf{L}_\mathcal{R}$–consistent, which is impossible by assumption. Therefore, φ is $\mathsf{L}_\mathcal{R}$-consistent.

- The proof that α is $\mathsf{L}_\mathcal{R}$–consistent is similar. Assume that it is not, that is, $\alpha \vdash_{\mathsf{L}_\mathcal{R}}$. Then, we have the following reasoning:

$$\dfrac{\dfrac{\dfrac{\alpha \vdash_{\mathsf{L}_\mathcal{R}}}{\alpha \vdash_{\mathsf{L}_\mathcal{R}} \bot \subset \varphi}\ C_R \qquad \dfrac{\varphi \vdash_{\mathsf{L}_\mathcal{R}} \varphi \quad \bot \vdash_{\mathsf{L}_\mathcal{R}}}{\varphi; \bot \subset \varphi \vdash_{\mathsf{L}_\mathcal{R}}}\ \supset_L}{\varphi; \alpha \vdash_{\mathsf{L}_\mathcal{R}}}\ Cut}{\varphi \circ \alpha \vdash_{\mathsf{L}_\mathcal{R}}}\ \circ_L$$

  As a consequence, we obtain that $\varphi \circ \alpha$ is not $\mathsf{L}_\mathcal{R}$–consistent, which is impossible by assumption. Therefore, α is $\mathsf{L}_\mathcal{R}$-consistent. □

Then, we have the following Lindenbaum-like lemma:

Lemma 9. *Any $\mathsf{L}_\mathcal{R}$–consistent set over $\mathcal{S}^\varphi$ ($\mathcal{S}^\alpha$) can be extended into a* maximal *$\mathsf{L}_\mathcal{R}$–consistent set over $\mathcal{S}^\varphi$ (resp. $\mathcal{S}^\alpha$).*

Proof. The proof follows the same reasoning as the proof of Lemma 4. □

The completeness proof. We are ready to prove the completeness of $\mathsf{L}_{\mathcal{R}}$. First, we define the canonical update model associated to $\mathsf{L}_{\mathcal{R}}$.

Definition 22 (Canonical update model)**.** The *canonical update model associated to* $\mathsf{L}_{\mathcal{R}}$ is the $\mathsf{L}_{\mathcal{R}}$-model $\mathcal{M}_c := (S^c, R_1^c, \dots R_m^c, \mathcal{R}_\supset, \mathcal{R}_\subset, \mathcal{R}_\circ, I^c)$ defined as follows:

- S^c is the disjoint union of the set $\mathcal{C}^\varphi$ of all maximal $\mathsf{L}_{\mathcal{R}}$–consistent sets of $\mathcal{S}^\varphi$ and the set $\mathcal{C}^\alpha$ of all maximal $\mathsf{L}_{\mathcal{R}}$–consistent sets of $\mathcal{S}^\alpha$;
- for all $\Gamma, \Gamma', \Gamma^f \in \mathcal{C}^\varphi$, all $\Gamma^\alpha \in \mathcal{C}^\alpha$, all $j = 1, \dots, m$,

$$\begin{array}{lll} (\Gamma, \Gamma') \in R_j^c & \text{iff} & \text{for all } \Box_j\varphi \in \Gamma, \text{ we have that } \varphi \in \Gamma' \\ (\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}_\supset & \text{iff} & \text{for all } \alpha \supset \varphi \in \Gamma, \text{ if } \alpha \in \Gamma^\alpha \text{ then } \varphi \in \Gamma^f \\ (\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}_\subset & \text{iff} & \text{for all } \varphi_f \subset \varphi \in \Gamma^\alpha, \text{if } \varphi \in \Gamma \text{ then } \varphi_f \in \Gamma^f \\ (\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}_\circ & \text{iff} & \text{for all } \varphi \in \Gamma \text{ and all } \alpha \in \Gamma^\alpha, \varphi \circ \alpha \in \Gamma^f \end{array}$$

- for all $p \in ATM$, all $\psi \in \mathcal{L}$,

$$\begin{array}{lll} p \in \mathcal{I}_c(\Gamma) & \text{iff} & p \in \Gamma \\ p_\psi \in \mathcal{I}_c(\Gamma^\alpha) & \text{iff} & p_\psi \in \Gamma^\alpha \end{array}$$

□

Lemma 10. *It holds that* $\mathcal{R}_\supset = \mathcal{R}_\subset = \mathcal{R}_\circ$.

Proof. The proof is similar to the proof of Lemma 11.25 of (Restall, 2000). □

Hence, from now on, we will denote this relation $\mathcal{R} := \mathcal{R}_\supset = \mathcal{R}_\subset = \mathcal{R}_\circ$.

Lemma 11 (Truth lemma)**.** *For all $\mathcal{L}_{\mathcal{R}}$-structure X, for all maximal consistent set Γ, we have that*

$$\mathcal{M}^c, \Gamma \Vdash X \quad \textit{iff} \quad X \in \Gamma. \tag{20}$$

Proof. The proof is by induction on the $\mathcal{L}_{\mathcal{R}}$–structure X. We start with the base case where $X := \varphi$ is a formula of $\mathcal{L}_{\mathcal{R}}$. Then, we will deal with the cases where X is of the form X, Y or $X; X^\alpha$.

The boolean cases are dealt with the same reasoning as in the previous section for modal logic. So, we only deal with the new connectives (we also recall the case for the modal operator).

- $\varphi := \Box_j\psi$: assume that $\mathcal{M}_c, \Gamma \models \Box_j\psi$. We are going to show that $\Box_j\psi \in \Gamma$. Let $S := \{\neg\psi\} \cup \{\varphi : \Box_j\varphi \in \Gamma\}$ and assume that S satisfies (1). Then, S can be extended to a maximal $\mathsf{L}_{\mathcal{R}}$–consistent set S^+ by Lemma 9. Now, $\psi \notin S^+$ because $\neg\psi \in S$. So, $\mathcal{M}_c, S^+ \models \neg\psi$. Moreover, $S^+ \in R_j(\Gamma)$ by definition of R_j. Hence, $\mathcal{M}_c, \Gamma \models \Diamond_j\neg\psi$. This is impossible by assumption. So, S is not $\mathsf{L}_{\mathcal{R}}$–consistent. So, there are $\varphi_1, \dots, \varphi_n \in S$ such that $\neg\psi, \varphi_1, \dots, \varphi_n \vdash_{\mathsf{L}_{\mathcal{R}}}$. Then, $\varphi_1, \dots, \varphi_n \vdash_{\mathsf{L}_{\mathcal{R}}} \neg\neg\psi$ by Rule $\neg_L$. So, by Rule k, $\Box_j\varphi_1, \dots, \Box_j\varphi_n \vdash_{\mathsf{L}_{\mathcal{R}}} \Box_j\neg\neg\psi$.

Moreover, $\Box_j\varphi_1, \ldots, \Box_j\varphi_n \in \Gamma$. So, because Γ is a maximal consistent set of $\mathcal{L}$, $\Box_j\neg\neg\psi \in \Gamma$ by Lemma 8. Moreover, $\neg\neg\psi \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \psi$, so $\Box_j\neg\neg\psi \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \Box_j\psi$ by Rule k. So, again by application of Lemma 8, $\Box_j\psi \in \Gamma$.

Assume that $\Box_j\psi \in \Gamma$. Then, for all Γ' such that $\Gamma' \in R_j(\Gamma)$, we have that $\psi \in \Gamma'$. Therefore, $\mathcal{M}_c, \Gamma' \models \psi$. So, $\mathcal{M}_c, \Gamma \models \Box_j\psi$ by definition.

- $\varphi := \alpha \supset \psi$:

Assume that $\alpha \supset \psi \in \Gamma$. Then, for all Γ^α, Γ^f such that $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$, if $\alpha \in \Gamma^\alpha$ then $\psi \in \Gamma^f$. That is, for all Γ^α, Γ^f such that $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$, if $\mathcal{M}_c, \Gamma^\alpha \Vdash \alpha$ then $\mathcal{M}_c, \Gamma^f \Vdash \psi$ by induction hypothesis. That is, $\mathcal{M}_c, \Gamma \Vdash \alpha \supset \psi$.

Assume that $\mathcal{M}_c, \Gamma \Vdash \alpha \supset \psi$ and assume towards a contradiction that $\alpha \supset \psi \notin \Gamma$.

1. Assume that $S := \{\varphi : \alpha \supset \varphi \in \Gamma\} \cup \{\neg\psi\}$ is not $\mathsf{L}_{\mathcal{R}}$–consistent. Then, there are $\varphi_1, \ldots, \varphi_n \in S$ such that $\neg\psi, \varphi_1, \ldots, \varphi_n \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}}$. Hence, $\varphi_1, \ldots, \varphi_n \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \neg\neg\psi$. So, because $\alpha \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \alpha$, we have that $\alpha \supset \varphi_1, \ldots, \alpha \supset \varphi_n; \alpha \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \neg\neg\psi$ by iterated applications of Rules $\supset_L$ and $(,/;)$ of Expression (15). Then, $\alpha \supset \varphi_1, \ldots, \alpha \supset \varphi_n \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \alpha \supset \neg\neg\psi$ $(*)$ by Rule $\supset_R$. Then, again by application of Lemma 8 to $(*)$, we have that $\alpha \supset \neg\neg\psi \in \Gamma^f$. Therefore, because $\alpha \supset \neg\neg\psi \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \alpha \supset \psi$, we have by application of Lemma 8 that $\alpha \supset \psi \in \Gamma^f$. This is impossible. Therefore, S is $\mathsf{L}_{\mathcal{R}}$–consistent and by Lemma 9 it can be extended into a maximal consistent set called Γ^f.

2. Now, let $S_\alpha := \{\alpha\} \cup \{\neg\beta : \text{there is } \psi \notin \Gamma^f, \beta \supset \psi \in \Gamma\}$. Assume that S_α is not $\mathsf{L}_{\mathcal{R}}$–consistent. Then, there are $\neg\beta_1, \ldots, \neg\beta_n \in S_\alpha$ such that $\alpha, \neg\beta_1, \ldots, \neg\beta_n \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}}$. So, $\alpha \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \beta_1, \ldots, \beta_n$ by $\neg_R$ and the Cut rule. Therefore, $\alpha \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \beta_1 \vee \ldots \vee \beta_n$. Let $\beta_1, \ldots, \beta_n$ be the formulas of Γ^f associated to $\psi_1, \ldots, \psi_n$. Then, $\bigvee_i \psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \bigvee_i \psi_i$. Then, by application of $\supset_L$, we have that $(\beta_1 \vee \ldots \vee \beta_n) \supset \bigvee_i \psi_i; \alpha \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \bigvee_i \psi_i$. So, by application of $\supset_R$ we have that $(\beta_1 \vee \ldots \vee \beta_n) \supset \bigvee_i \psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \alpha \supset \bigvee_i \psi_i$ $(*)$. However, for all $i \in \{1, \ldots, k\}$, $\beta_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \beta_i$ and $\psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \bigvee_i \psi_i$, so by application of $\supset_L$, we have that $\beta_i \supset \psi_i; \beta_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \bigvee_i \psi_i$, for all $i \in \{1, \ldots, k\}$. Therefore, by $\supset_R$, we have that $\beta_i \supset \psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \beta_i \supset \bigvee_i \psi_i$. Now, by Expression (18) of Fact 4, we have that $\beta_1 \supset \bigvee_i \psi_i, \ldots, \beta_n \supset \bigvee_i \psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} (\beta_1 \vee \ldots \vee \beta_n) \supset \bigvee_i \psi_i$ $(**)$. So, by the Cut rule applied to $(*)$ and $(**)$, we have that $\beta_1 \supset \bigvee_i \psi_i, \ldots, \beta_n \supset \bigvee_i \psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \alpha \supset \bigvee_i \psi_i$. Therefore, by iterated application of Lemma 8, we have that $\alpha \supset \bigvee_i \psi_i \in \Gamma$, because $\beta_1 \supset \bigvee_i \psi_i, \ldots, \beta_n \supset \bigvee_i \psi_i \in \Gamma$. Hence, $\bigvee_i \psi_i \in S$ by definition of S. So, $\bigvee_i \psi_i \in \Gamma^f$ by definition of Γ^f. However, for all i, $\psi_i \notin \Gamma^f$. But for all i, $\psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}} \psi_i$, so $\psi_i, \neg\psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}}$ by $\neg_L$ and $\psi_i, \bigwedge_i \neg\psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}}$ by $\wedge_L$. So, $\bigvee_i \psi_i, \bigwedge_i \neg\psi_i \mathrel{|\!\!\frac{}{\mathsf{L}_{\mathcal{R}}}}$ by $\vee_L$ and W_L. But for all i, $\neg\psi_i \in \Gamma^f$, so $\bigwedge_i \neg\psi_i \in \Gamma_f$ and $\bigvee_i \psi_i \in \Gamma^f$. This is impossible. Therefore, S_α is

$\mathsf{L}_{\mathcal{R}}$–consistent. So, it can be extended into a maximal $\mathsf{L}_{\mathcal{R}}$–consistent set called Γ^α.

Finally, we need to check that $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$. Let $\beta \supset \varphi \in \Gamma$ and assume that $\beta \in \Gamma^\alpha$ but $\varphi \notin \Gamma^f$. Then, by definition of Γ^α, $\neg\beta \in \Gamma^\alpha$, which contradicts the fact that $\beta \in \Gamma^\alpha$. So, $\varphi \in \Gamma^f$. Hence, there are $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$ such that $\alpha \in \Gamma^\alpha$ and $\neg\psi \in \Gamma^f$. So, there are $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$ such that $\mathcal{M}_c, \Gamma^\alpha \Vdash \alpha$ and $\mathcal{M}_c, \Gamma^f \Vdash \neg\psi$ by Induction Hypothesis. Therefore, $\mathcal{M}_c, \Gamma \nvDash \alpha \supset \psi$, which contradicts our assumption. We have reached a contradiction, so, finally, $\alpha \supset \psi \in \Gamma$.

- $\varphi := \psi \subset \chi$:

  Assume that $\psi \subset \chi \in \Gamma^\alpha$. Then, for all $\Gamma, \Gamma^f \in \mathcal{M}_c$ such that $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$, if $\chi \in \Gamma$ then $\psi \in \Gamma^f$. Then, for all $\Gamma, \Gamma^f \in \mathcal{M}_c$ such that $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$, if $\mathcal{M}_c, \Gamma \Vdash \chi$ then $\mathcal{M}_c, \Gamma^f \Vdash \psi$. That is, $\mathcal{M}_c, \Gamma^\alpha \Vdash \psi \subset \chi$.

  Assume that $\mathcal{M}_c, \Gamma^\alpha \Vdash \psi \subset \chi$. Assume towards a contradiction that $\psi \subset \chi \notin \Gamma^\alpha$.

  1. Let $S_f := \{\neg\psi\} \cup \{\varphi : \varphi \subset \chi \in \Gamma^\alpha\}$ and assume that S_f is not $\mathsf{L}_{\mathcal{R}}$–consistent. Then, there are $\varphi_1, \ldots, \varphi_n \in S_f$ such that $\neg\psi, \varphi_1, \ldots, \varphi_n \models_{\mathsf{L}_{\mathcal{R}}}$. So, because $\chi \models_{\mathsf{L}_{\mathcal{R}}} \chi$, by iterated application of Rules $\subset_L$ and $(;/,)$ of Expression (14), we have that $\chi; (\varphi_1 \subset \chi, \ldots, \varphi_n \subset \chi) \models_{\mathsf{L}_{\mathcal{R}}} \psi$. Hence, $\varphi_1 \subset \chi, \ldots, \varphi_n \subset \chi \models_{\mathsf{L}_{\mathcal{R}}} \psi \subset \chi$ by Rule $\subset_R$. But $\varphi_1 \subset \chi, \ldots, \varphi_n \subset \chi \in \Gamma^\alpha$, so $\psi \subset \chi \in \Gamma^\alpha$ again by application of Lemma 8. This is impossible by assumption. Therefore, S_f is $\mathsf{L}_{\mathcal{R}}$–consistent. Then, S_f can be extended into a maximal $\mathsf{L}_{\mathcal{R}}$–consistent set $\Gamma^f \in \mathcal{M}_c$ by Lemma 9.

  2. Now, let $S := \{\chi\} \cup \{\neg\varphi : \text{there is } \psi \notin \Gamma^f \text{ such that } \psi \subset \varphi \in \Gamma^\alpha\}$. Assume that S is not $\mathsf{L}_{\mathcal{R}}$–consistent. Then, there are $\neg\varphi_1, \ldots, \neg\varphi_n \in S$ such that $\chi, \neg\varphi_1, \ldots, \neg\varphi_n \models_{\mathsf{L}_{\mathcal{R}}}$. Then, $\chi \models_{\mathsf{L}_{\mathcal{R}}} \varphi_1, \ldots, \varphi_n$ by Rule $\neg_L$ and the Cut Rule. So, $\chi \models_{\mathsf{L}_{\mathcal{R}}} \varphi_1 \vee \ldots \vee \varphi_n$. Let $\psi_1, \ldots, \psi_n$ be the formulas of Γ^f associated to $\varphi_1, \ldots, \varphi_n$ through S. Then, because $\bigvee_i \psi_i \models_{\mathsf{L}_{\mathcal{R}}} \bigvee_i \psi_i$, we have that $\chi; \bigvee_i \psi_i \subset (\varphi_1 \vee \ldots \vee \varphi_n) \models_{\mathsf{L}_{\mathcal{R}}} \bigvee_i \psi_i$ by Rule $\subset_L$. Therefore, $\bigvee_i \psi_i \subset (\varphi_1 \vee \ldots \vee \varphi_n) \models_{\mathsf{L}_{\mathcal{R}}} \bigvee_i \psi_i \subset \chi$ $(*)$ by Rule $\subset_R$. However, we have that $\bigvee_i \psi_i \subset \varphi_i \in \Gamma^\alpha$ for all i and, by Expression (19) of Fact 4, $\bigvee_i \psi_i \subset \varphi_1, \ldots, \bigvee_i \psi_i \subset \varphi_n \models_{\mathsf{L}_{\mathcal{R}}} \bigvee_i \psi_i \subset (\varphi_1 \vee \ldots \vee \varphi_n)$. Therefore, by application of Lemma 8, we have that $\bigvee_i \psi_i \subset (\varphi_1 \vee \ldots \vee \varphi_n) \in \Gamma^\alpha$. Then, by application of Lemma 8 to $(*)$, we have that $\bigvee_i \psi_i \subset \chi \in \Gamma^\alpha$. Then, by definition of S_f, we have that $\bigvee_i \psi_i \in S_f$. So, $\bigvee_i \psi_i \in \Gamma^f$. However, $\psi_i \notin \Gamma^f$ for all i. Therefore, $\bigwedge_i \neg\psi_i \in \Gamma^f$, contradicting $\bigvee_i \psi_i \in \Gamma^f$. Hence, we reach a contradiction. Therefore, S is $\mathsf{L}_{\mathcal{R}}$–consistent. So, it can be extended to a maximal $\mathsf{L}_{\mathcal{R}}$–consistent set $\Gamma \in \mathcal{M}_c$ by Lemma 9.

  Now, we prove that we have $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$. Let $\psi \subset \chi \in \Gamma^\alpha$ and assume towards a contradiction that $\chi \in \Gamma$ but $\psi \notin \Gamma^f$. Then, $\neg\chi \in S$ by definition of S, so $\neg\chi \in \Gamma$. This is impossible because $\chi \in \Gamma$ and Γ is a maximal $\mathsf{L}_{\mathcal{R}}$–consistent set.

So, if $\chi \in \Gamma$ then $\psi \in \Gamma^f$. So, for all $\psi \subset \chi \in \Gamma^\alpha$, if $\chi \in \Gamma$ then $\psi \in \Gamma^f$. Hence, $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$.

Moreover, $\chi \in \Gamma$ and $\neg\psi \in \Gamma^f$, so by Induction Hypothesis, $\mathcal{M}_c, \Gamma \Vdash \chi$ and not $\mathcal{M}_c, \Gamma^f \Vdash \psi$. Therefore, we do not have that $\mathcal{M}_c, \Gamma^\alpha \Vdash \psi \subset \chi$, which is impossible by assumption. So, finally, $\psi \subset \chi \in \Gamma^\alpha$.

- $\varphi := \psi \circ \alpha$:

  Assume that $\varphi \circ \alpha \in \Gamma^f$. We must show that $\mathcal{M}_c, \Gamma^f \Vdash \varphi \circ \alpha$, that is, there are $\Gamma, \Gamma^\alpha \in \mathcal{M}_c$ such that $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$, $\mathcal{M}_c, \Gamma \Vdash \varphi$ and $\mathcal{M}_c, \Gamma^\alpha \Vdash \alpha$, *i.e.*, there are $\Gamma, \Gamma^\alpha \in \mathcal{M}_c$ such that $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$, $\varphi \in \Gamma$ and $\alpha \in \Gamma^\alpha$. We construct the maximal consistent sets Γ and Γ^α following the steps described in the 'pseudo'-Algorithm 1 (we call it '*pseudo*'-Algorithm because it is not terminating, we only introduce it in order to better explain the way we construct Γ and Γ^α).

  We prove that the 'pseudo'-Algorithm 1 is well-defined. To do so, we prove that for all $n \in \mathbb{N}$,

$$\begin{array}{rcll} (\bigwedge \Gamma_n \wedge \varphi_n) & \circ & (\bigwedge \Gamma_n^\alpha \wedge \alpha_n) \in \Gamma^f & \text{or} \\ (\bigwedge \Gamma_n \wedge \neg\varphi_n) & \circ & (\bigwedge \Gamma_n^\alpha \wedge \alpha_n) \in \Gamma^f & \text{or} \\ (\bigwedge \Gamma_n \wedge \neg\varphi_n) & \circ & (\bigwedge \Gamma_n^\alpha \wedge \neg\alpha_n) \in \Gamma^f & \text{or} \\ (\bigwedge \Gamma_n \wedge \varphi_n) & \circ & (\bigwedge \Gamma_n^\alpha \wedge \neg\alpha_n) \in \Gamma^f & \end{array} \tag{21}$$

  (The "or" is inclusive.) Expression (21) is due to the fact that for all $\varphi, \psi \in \mathcal{L}_\mathcal{R}^\varphi$, all $\alpha, \beta \in \mathcal{L}_\mathcal{R}^\alpha$, we can prove the following:

$$\begin{aligned} \psi \circ \beta \vdash_{\mathsf{L}_\mathcal{R}} & (\psi \wedge \varphi) \circ (\beta \wedge \alpha) \vee (\psi \wedge \neg\varphi) \circ (\beta \wedge \alpha) \vee \\ & (\psi \wedge \neg\varphi) \circ (\beta \wedge \neg\alpha) \vee (\psi \wedge \varphi) \circ (\beta \wedge \neg\alpha) \end{aligned} \tag{22}$$

  To prove Expression (22), we use Expression (16) of Fact 4 and the fact that $\psi \circ \beta \vdash_{\mathsf{L}_\mathcal{R}} ((\psi \wedge \varphi) \vee (\psi \wedge \neg\varphi)) \circ ((\beta \wedge \alpha) \vee (\beta \wedge \neg\alpha))$, which is itself proved by application of rules $\circ_R$ and then $\circ_L$ to $\psi \vdash_{\mathsf{L}_\mathcal{R}} (\psi \wedge \varphi) \vee (\psi \wedge \neg\varphi)$ and $\beta \vdash_{\mathsf{L}_\mathcal{R}} (\beta \wedge \alpha) \vee (\beta \wedge \neg\alpha)$. Replacing in Expression (22) ψ with $\bigwedge \Gamma_n$, β with $\bigwedge \Gamma_n^\alpha$, φ with φ_n and α with α_n, and using Lemma 8 together with the fact that $\bigwedge \Gamma_n \circ \bigwedge \Gamma_n^\alpha \in \Gamma^f$, we obtain the result of Expression (21), because Γ^f is a maximal consistent set. So, the 'pseudo'-algorithm is well-defined.

  Now, we prove that Γ and Γ^α are maximal consistent sets of $\mathcal{S}^\varphi$ and $\mathcal{S}^\alpha$ respectively. By Fact 5, for all $n \in \mathbb{N}$, Γ_n and Γ_n^α are $\mathsf{L}_\mathcal{R}$-consistent, because $\bigwedge \Gamma_n \circ \bigwedge \Gamma_n^\alpha$ is $\mathsf{L}_\mathcal{R}$-consistent since $\bigwedge \Gamma_n \circ \bigwedge \Gamma_n^\alpha \in \Gamma^f$ and Γ^f is $\mathsf{L}_\mathcal{R}$-consistent. So, at line 22 of the 'pseudo'-algorithm, Γ_0 and Γ_0^α are also $\mathsf{L}_\mathcal{R}$-consistent, since otherwise there would be a n such that Γ_n and Γ_n^α are not $\mathsf{L}_\mathcal{R}$-consistent. Hence, Γ and Γ^α are also $\mathsf{L}_\mathcal{R}$-consistent at the end of the 'pseudo'-algorithm by definition of the rest of the 'pseudo'-algorithm. Moreover, by construction of Γ and Γ^α, because all pairs of $\mathcal{S}^\varphi \times \mathcal{S}^\alpha$ are enumerated, Γ and Γ^α are *maximal* consistent sets of $\mathcal{S}^\varphi$ and $\mathcal{S}^\alpha$ respectively.

Algorithm 1

Require: $(\varphi, \alpha) \in \mathcal{L}_{\mathcal{R}}^{\varphi} \times \mathcal{L}_{\mathcal{R}}^{\alpha}$ and a maximal $\mathsf{L}_{\mathcal{R}}$–consistent set Γ^f of $\mathcal{L}_{\mathcal{R}}^{\varphi}$.
Ensure: A pair of maximal $\mathsf{L}_{\mathcal{R}}$-consistent sets $(\Gamma, \Gamma^{\alpha})$ such that $(\Gamma, \Gamma^{\alpha}, \Gamma^f) \in \mathcal{R}$, $\varphi \in \Gamma$ and $\alpha \in \Gamma^{\alpha}$.

Let $(\varphi_0, \alpha_0), \ldots, (\varphi_n, \alpha_n), \ldots$ be an enumeration of $\mathcal{L}_{\mathcal{R}}^{\varphi} \times \mathcal{L}_{\mathcal{R}}^{\alpha}$ and let $(X_0, X_0^{\alpha}), \ldots, (X_n, X_n^{\alpha}), \ldots$ be an enumeration of $\mathcal{S}^{\varphi} \times \mathcal{S}^{\alpha} - \mathcal{L}_{\mathcal{R}}^{\varphi} \times \mathcal{L}_{\mathcal{R}}^{\alpha}$

$\Gamma_0 := \{\varphi\}$
$\Gamma_0^{\alpha} := \{\alpha\}$

For all $n \geq 0$ **do**
if $(\bigwedge \Gamma_n \wedge \varphi_n) \circ (\bigwedge \Gamma_n^{\alpha} \wedge \alpha_n) \in \Gamma^f$ **then**
  $\Gamma_{n+1} := \Gamma_n \cup \{\varphi_n\}$
  $\Gamma_{n+1}^{\alpha} := \Gamma_n^{\alpha} \cup \{\alpha_n\}$
else if $(\bigwedge \Gamma_n \wedge \neg\varphi_n) \circ (\bigwedge \Gamma_n^{\alpha} \wedge \alpha_n) \in \Gamma^f$ **then**
  $\Gamma_{n+1} := \Gamma_n \cup \{\neg\varphi_n\}$
  $\Gamma_{n+1}^{\alpha} := \Gamma_n^{\alpha} \cup \{\alpha_n\}$
else if $(\bigwedge \Gamma_n \wedge \neg\varphi_n) \circ (\bigwedge \Gamma_n^{\alpha} \wedge \neg\alpha_n) \in \Gamma^f$ **then**
  $\Gamma_{n+1} := \Gamma_n \cup \{\neg\varphi_n\}$
  $\Gamma_{n+1}^{\alpha} := \Gamma_n^{\alpha} \cup \{\neg\alpha_n\}$
else
  $\Gamma_{n+1} := \Gamma_n \cup \{\varphi_n\}$
  $\Gamma_{n+1}^{\alpha} := \Gamma_n^{\alpha} \cup \{\neg\alpha_n\}$
end if

$\Gamma_0 := \bigcup_{n \geq 0} \Gamma_n$
$\Gamma_0^{\alpha} := \bigcup_{n \geq 0} \Gamma_n^{\alpha}$

For all $n \geq 0$ **do**
if $\Gamma_n \cup \{X_n\}$ is $\mathsf{L}_{\mathcal{R}}$-consistent **then**
  $\Gamma_{n+1} := \Gamma_n \cup \{X_n\}$
end if
if $\Gamma_n^{\alpha} \cup \{X_n^{\alpha}\}$ is $\mathsf{L}_{\mathcal{R}}$-consistent **then**
  $\Gamma_{n+1}^{\alpha} := \Gamma_n^{\alpha} \cup \{X_n\}$
end if

$\Gamma := \bigcup_{n \geq 0} \Gamma_n$
$\Gamma^{\alpha} := \bigcup_{n \geq 0} \Gamma_n^{\alpha}$

Finally, we need to prove that $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$. To do so, it suffices to prove that for all $\varphi \in \Gamma$, all $\alpha \in \Gamma^\alpha$, $\varphi \circ \alpha \in \Gamma^f$ by Definition 22 of the canonical model. Let $\varphi \in \Gamma$ and let $\alpha \in \Gamma^\alpha$. Then, there is $n \in \mathbb{N}$ such that $(\varphi, \alpha) = (\varphi_n, \alpha_n)$. Then, by definition of the algorithm, we must have that $(\Gamma_n \wedge \varphi) \circ (\Gamma_n^\alpha \wedge \alpha) \in \Gamma^f$. Therefore, because $\Gamma_n \wedge \varphi \vdash_{\mathsf{L}_\mathcal{R}} \varphi$ and $\Gamma_n^\alpha \wedge \alpha \vdash_{\mathsf{L}_\mathcal{R}} \alpha$, we have that $(\Gamma_n \wedge \varphi) \circ (\Gamma_n^\alpha \wedge \alpha) \vdash_{\mathsf{L}_\mathcal{R}} \varphi \circ \alpha$ by application of rules $\circ_R$ and then $\circ_L$. Thus, $\varphi \circ \alpha \in \Gamma^f$ by Lemma 8, and therefore $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$.

Now, we prove the converse, that is, if $\mathcal{M}_c, \Gamma^f \Vdash \varphi \circ \alpha$, then $\varphi \circ \alpha \in \Gamma^f$. By definition of $\circ$, we have that there are Γ and Γ^α such that $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$, $\mathcal{M}_c, \Gamma \Vdash \varphi$ and $\mathcal{M}_c, \Gamma^\alpha \Vdash \alpha$. So, by Induction Hypothesis, we have that $\varphi \in \Gamma$ and $\alpha \in \Gamma^\alpha$. So, by definition of $\mathcal{R}(= \mathcal{R}_\circ)$, we must have that $\varphi \circ \alpha \in \Gamma^f$, since $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$. This completes the proof.

Finally, we deal with the case where X is a $\mathcal{L}_\mathcal{R}$–structure of the form X, Y or $X; X^\alpha$. These two cases are proved straightforwardly because $\mathcal{M}_c, \Gamma \Vdash X$ iff $\mathcal{M}_c, \Gamma \Vdash t(X)$ by Expression (12) of Fact 3, because $\mathcal{M}_c, \Gamma \Vdash t(X)$ iff $t(X) \in \Gamma$ by the preceding reasoning (since $t(X) \in \mathcal{L}_\mathcal{R}$), and finally because $t(X) \in \Gamma$ iff $X \in \Gamma$ by Expression (13) of Fact 3. □

Proof of Theorem 7. The proof of soundness is routine, so we only prove completeness, *i.e.*, we prove that for all $\mathcal{L}_\mathcal{R}$–sequent $X \vdash Y$, if $X \Vdash Y$ holds then $X \vdash_{\mathsf{L}_\mathcal{R}} Y$ holds. Assume towards a contradiction that $X \Vdash Y$ and that it is not the case that $X \vdash_{\mathsf{L}_\mathcal{R}} Y$. However, we have that $X, \neg Y \vdash_{\mathsf{L}_\mathcal{R}}$ iff $X \vdash_{\mathsf{L}_\mathcal{R}} Y$ by application of $\neg_L$ (from right to left) and by application of $\neg_R$, the Cut rule and the fact that $\neg\neg Y \vdash_{\mathsf{L}_\mathcal{R}} Y$ (from left to right). Then, the set $\{X, \neg Y\}$ is $\mathsf{L}_\mathcal{R}$-consistent. So, by Lemma 9, it can be extended into a maximal $\mathsf{L}_\mathcal{R}$-consistent set Γ. Then, by the Truth Lemma 11, we have that $\mathcal{M}_c, \Gamma \models X, \neg Y$. Hence, it is not the case that $X \Vdash Y$. This contradicts our assumption and this completes the proof. □

5 Case Study: the DEL Product Update

So far, we did not impose any restriction on our ternary relation. Therefore, there is no reason that it corresponds to the DEL product update of Definition 7 since this update was of a very specific kind: it was deterministic, partial and bisimulation invariant. In this section, we are going to investigate which axioms and inference rules need to be added in order to recover our previous definition of the DEL product update. Doing so, we will provide a new axiomatization of the DEL product update which is different and more modular than the one proposed in (Aucher, 2011). Also, we will show that the operators of progression, regression and epistemic planning introduced in (Aucher, 2011, 2012) are in fact specific instances of the standard operators of substructural logic.

5.1 Towards a Correspondence Theory

The central object of our semantics is a ternary relation representing an update. Thanks to our logical language, we can now elicit a number of axioms and inference rules that define specific properties of this ternary relation. In other words, we can develop a genuine correspondence theory for the notion of update.

Definition 23. Below, p ranges over ATM, φ, ψ, χ range over $\mathcal{L}^{\varphi}_{\mathcal{R}}$ and α over $\mathcal{L}^{\alpha}_{\mathcal{R}}$.

$$p;\alpha \vdash p \qquad \neg p;\alpha \vdash \neg p \qquad (\textit{Atom–p \& Atom–}\neg p)$$

$$\neg\psi; p_\psi \vdash \qquad (\textit{Precondition})$$

$$\Box_j\varphi \circ \Box_j\alpha \vdash \Box_j(\varphi \circ \alpha) \qquad (\textit{Back–update})$$

$$\frac{\varphi;\alpha \vdash \varphi_f}{\Diamond_j(\varphi \wedge \chi); \Diamond_j(\alpha \wedge p_\chi) \vdash \Diamond_j\varphi_f} \qquad (\textit{Forth–update})$$

□

(*Atom–p & Atom–¬p*) illustrate the fact that we deal as in the standard framework of DEL with epistemic events, i.e. events which do not change atomic facts. Axiom schema (*Precondition*) illustrates the fact that an atomic event can occur only in a possible world where its precondition holds. The reading of Axiom schema (*Back–update*) is as follows: if the current situation results from the occurrence of an event during which j believed that α in a situation where j believed that φ, then in this current situation, j believes that it results from the occurrence of an event satisfying α in an initial situation satisfying φ. As for Rule (*Forth–update*), it turns out that the informal motivations for the definition of the DEL product update by Baltag and Moss (2004) are somehow formalized by Rule (*Forth–update*). Here is how the product update was informally motivated in this paper (the notations in this quotation are replaced by our notations):

> "The update product restricts the full Cartesian product $W \times W^\alpha$ to the smaller set $W \otimes W^\alpha$ in order to insure that states *survive* actions in the appropriate sense. [...] The components of our $\mathcal{L}_\alpha$-models are "simple actions", so the uncertainty regarding the action is assumed to be independent of the uncertainty regarding the current (input) state. This independence allows us to "multiply" these two uncertainties in order to compute the uncertainty regarding the output state: if whenever the input state is w, agent j thinks the input might be some other state v, and if whenever the current action happening is e, agent j thinks the current action might be some other action f, and if v survives f, then whenever the output state (w, e) is reached, agent j thinks the alternative output state (v, f) might have been reached." (Baltag and Moss, 2004, p. 194)

Now, if one thinks of formulas φ, α and φ_f in Rule (*Forth–update*) as representing respectively the input state v, the action f and the output state (v, f), then the conclusion of this rule somehow formalizes these informal motivations.

Note that the Axiom schemata (*Atom–p & Atom–¬p*) and (*Precondition*) correspond to the Axiom schemata $\mathsf{A_4}, \mathsf{A_5}$ and $\mathsf{A_6}$ of (Aucher, 2011) respectively, and Rule (*Forth–update*) corresponds to Rule $\mathsf{R_5}$ of (Aucher, 2011). Rule $\mathsf{R_4}$ of (Aucher, 2011) is derivable from Axiom schema (*Back–update*) and the modal rule k:

$$\frac{\varphi; \alpha \vdash \varphi_f}{\Box_j \varphi; \Box_j \alpha \vdash \Box_j \varphi_f} \ \mathsf{R_4}$$

In fact, we can even prove in $\mathsf{L}_\mathcal{R}$ together with (*Back–update*) an even stronger inference rule which generalizes uniformly both Rules k and $\mathsf{R_4}$:

$$\frac{X \vdash \varphi}{\Box_j X \vdash \Box_j \varphi} \ k^+$$

where $\Box_j X$ is defined inductively as follows: $\Box_j X := \Box_j \varphi$ if $X = \varphi$, $\Box_j X := (\Box_j Y, \Box_j Z)$ if $X = (Y, Z)$ and $\Box_j X := (\Box_j Y; \Box_j Z)$ if $X = (Y; Z)$. Now, we define some conditions on update models that will correspond exactly to the validity of our above axioms and inference rules.

Definition 24. Let $\mathcal{M}_\mathcal{R} = (\mathcal{P}, \mathcal{R}_1, \ldots, \mathcal{R}_m, \mathcal{R}, \mathcal{I})$ be an update model. Then, $\mathcal{M}_\mathcal{R}$ satisfies the conditions listed on the right hand side below when for all $((\mathcal{M}, w), (\mathcal{E}, e), (\mathcal{M}_f, w_f)) \in \mathcal{R}$,

- for all $p \in ATM$, $\mathcal{M}, w \models p$ iff $\mathcal{M}_f, w_f \models p$ *(Atom–p & Atom–¬p)*
- $\mathcal{M}, w \models I^\alpha(e)$ *(Precondition)*
- for all $v_f \in \mathcal{R}_j(w_f)$, there are $v \in \mathcal{R}_j(w)$ and $f \in \mathcal{R}_j(e)$ such that $((\mathcal{M}, v), (\mathcal{E}, f), (\mathcal{M}_f, v_f)) \in \mathcal{R}$. *(Back–update)*
- for all $v \in \mathcal{R}_j(w)$, all $f \in \mathcal{R}_j(e)$, if $\mathcal{M}, v \models I^\alpha(f)$ then there is $v_f \in \mathcal{R}_j(w_f)$ such that $((\mathcal{M}, v), (\mathcal{E}, f), (\mathcal{M}_f, v_f)) \in \mathcal{R}$ *(Forth–update)*

$\Box$

Then, we have the following correspondence results:

Theorem 12 (Canonicity)**.** *Let $S \subseteq \{$(*Atom–p & Atom–¬p*), (*Precondition*), (*Back–update*), (*Forth–update*)$\}$ The sequent calculus $\mathsf{L}_\mathcal{R} + S$ is sound and complete w.r.t. the class of update models satisfying the corresponding conditions of S (given in Definition 24).*

Proof. Soundness is routine. We only prove completeness of (*Back–update*) and (*Forth–update*). The proofs for the other cases are without particular difficulty.

(*Back–update*): Given a $\mathsf{L}_\mathcal{R}$ + (*Back–update*)–consistent set Γ of $\mathcal{L}_\mathcal{R}$, it suffices to find an update model $\mathcal{M}_\mathcal{R}$ and $x \in \mathcal{M}_\mathcal{R}$ such that (1) $\mathcal{M}_\mathcal{R}, x \models \Gamma$ and (2) $\mathcal{M}_\mathcal{R}$ satisfies the condition (*Back–update*). Let $\mathcal{M}_c = (\mathcal{P}_c, \mathcal{R}_1, \ldots, \mathcal{R}_m, \mathcal{R}_c, \mathcal{I}_c)$ be the canonical model for $\mathsf{L}_\mathcal{R}$ + (*Back–update*) as defined in the proof of Theorem 7, and let Γ^+

be any $\mathsf{L}_{\mathcal{R}} + (\textit{Back–update})$-consistent maximal extension of Γ obtained by Lemma 9. Then, by Lemma 11, $\mathcal{M}_c, \Gamma^+ \Vdash \Gamma$, so step (1) is established. It remains to show that $\mathcal{M}_c$ satisfies condition (*Back–update*). Let $(\Gamma, \Gamma^\alpha, \Gamma^f) \in \mathcal{R}$ and let $\Gamma_1^f \in R_j(\Gamma^f)$. Let $(\varphi_0^1, \alpha_0^1), \ldots, (\varphi_n^1, \alpha_n^1), \ldots$ be a countable enumeration of $S := S^\varphi \times S^\alpha$, where $S^\varphi := \{\varphi \in \mathcal{L}_{\mathcal{R}}^\varphi : \Box_j \varphi \in \Gamma\}$ and $S^\alpha := \{\alpha \in \mathcal{L}_{\mathcal{R}}^\alpha : \Box_j \alpha \in \Gamma^\alpha\}$. We concatenate to this enumeration an arbitrary countable enumeration of $\mathcal{S}^\varphi \times \mathcal{S}^\alpha - S$. This yields a countable enumeration $(\varphi_0, \alpha_0), \ldots, (\varphi_n, \alpha_n), \ldots$ of $\mathcal{S}^\varphi \times \mathcal{S}^\alpha$. Then, we apply the 'pseudo'-Algorithm 1 with $\Gamma_0 := \emptyset$ and $\Gamma_0^\alpha := \emptyset$. This yields two sets Γ_1 and Γ_1^α. Because our enumeration starts with the formulas of S, we are sure to obtain that $S^\varphi \subseteq \Gamma_1$ and $S^\alpha \subseteq \Gamma_1^\alpha$. Indeed, because $\Box_j \varphi \circ \Box_j \alpha \in \Gamma^f$ for all $\varphi \in S^\varphi$ and $\alpha \in S^\alpha$, we must have that $\Box_j(\varphi \circ \alpha) \in \Gamma^f$ by Axiom (*Back–update*). Therefore, because $\Gamma_1^f \in R_j(\Gamma^f)$, we must have that $\varphi \circ \alpha \in \Gamma_1^f$. This explains that during the execution of the 'pseudo'-Algorithm 1, the first conditional will always be satisfied for the pairs of formulas of S since we start our enumeration with them. Therefore, we will have that $S^\varphi \subseteq \Gamma_1$ and $S^\alpha \subseteq \Gamma_1^\alpha$. So, by definition of R_j in $\mathcal{M}_c$, we have that $\Gamma_1 \in R_j(\Gamma)$ and $\Gamma_1^\alpha \in R_j(\Gamma^\alpha)$. Moreover, by the same argument as the one given after the 'pseudo'-Algorithm 1, we must have that $(\Gamma_1, \Gamma_1^\alpha, \Gamma_1^f) \in \mathcal{R}$. This proves that $\mathcal{M}_c$ satisfies the condition (*Back–update*).

(*Forth–update*): Given a $\mathsf{L}_{\mathcal{R}} + (\textit{Forth–update})$–consistent set Γ of $\mathcal{L}_{\mathcal{R}}$, it suffices to find an update model $\mathcal{M}_{\mathcal{R}}$ and $x \in \mathcal{M}_{\mathcal{R}}$ such that (1) $\mathcal{M}_{\mathcal{R}}, x \models \Gamma$ and (2) $\mathcal{M}_{\mathcal{R}}$ satisfies condition (*Forth–update*). Let $\mathcal{M}_c = (\mathcal{P}_c, \mathcal{R}_c, \mathcal{I}_c)$ be the canonical model for $\mathsf{L}_{\mathcal{R}} + (\textit{Forth–update})$ as defined in the proof of Theorem 7, and let Γ^+ be any $\mathsf{L}_{\mathcal{R}} + (\textit{Forth–update})$–consistent maximal extension of Γ. By Lemma 11, $\mathcal{M}_c, \Gamma^+ \Vdash \Gamma$ so step (1) is established. It remains to show that $\mathcal{M}_c$ satisfies condition (*Forth–update*). Let $(\Gamma, \Gamma^\alpha, \Gamma_f) \in \mathcal{R}_c$ and let $\Gamma_1 \in R_j(\Gamma)$, $\Gamma_1^\alpha \in \mathcal{R}_j(\Gamma^\alpha)$ such that $\mathcal{M}_c, \Gamma_1 \Vdash I^\alpha(\Gamma_1^\alpha)$. We are going to show that there is $\Gamma'_f \in \mathcal{R}_j(\Gamma_f)$ such that $(\Gamma_1, \Gamma_1^\alpha, \Gamma'_f) \in \mathcal{R}_c$. Let us consider the following set of formulas: $S := S_1 \cup S_2$, where $S_1 := \{\psi : \Box_j \psi \in \Gamma_f\}$ and $S_2 := \{\varphi \circ \alpha : \varphi \in \Gamma_1, \alpha \in \Gamma_1^\alpha\}$. We prove that S is $\mathsf{L}_{\mathcal{R}} + (\textit{Forth–update})$-consistent. Assume towards a contradiction that it is not. Then, there are $\psi_1, \ldots, \psi_m \in S_1$ and $\varphi_1 \circ \alpha_1, \ldots, \varphi_n \circ \alpha_n \in S_2$ such that $\psi_1, \ldots, \psi_m, \varphi_1 \circ \alpha_1, \ldots, \varphi_n \circ \alpha_n \vdash_{\mathsf{L}_{\mathcal{R}}}$. Then, $\varphi_1 \circ \alpha_1, \ldots, \varphi_n \circ \alpha_n \vdash_{\mathsf{L}_{\mathcal{R}}} \neg\psi_1, \ldots, \neg\psi_m$ by $\neg_R$. Therefore, by soundness of $\mathsf{L}_{\mathcal{R}}$, we have that $\varphi_1 \circ \alpha_1, \ldots, \varphi_n \circ \alpha_n, \Vdash \neg\psi_1, \ldots, \neg\psi_m$. Then, again by validity of the Cut Rule and Rule $\circ_L$, $(\varphi_1; \alpha_1), \ldots, (\varphi_n; \alpha_n) \Vdash \neg\psi_1, \ldots, \neg\psi_n$. Then, by soundness of Rules $\wedge_L$ and C_L, $(\varphi_1; \alpha_1 \wedge \ldots \wedge \alpha_n), \ldots, (\varphi_n; \alpha_1 \wedge \ldots \wedge \alpha_n) \Vdash \neg\psi_1, \ldots, \neg\psi_n$. So, by soundness of Rule $(,/;)$, $\varphi_1, \ldots, \varphi_n; \alpha_1 \wedge \ldots \wedge \alpha_n \Vdash \neg\psi_1, \ldots, \neg\psi_n$. Then, again by soundness of Rules $\wedge_L$ and C_L, C_R, we have that $\varphi_1 \wedge \ldots \wedge \varphi_n; \alpha_1 \wedge \ldots \wedge \alpha_n \Vdash \neg(\psi_1 \wedge \ldots \wedge \psi_n)$. So, for all $\psi \in \mathcal{L}_{\mathcal{R}}^\varphi$, $\Diamond_j(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \psi); \Diamond_j(\alpha_1 \wedge \ldots \wedge \alpha_n \wedge p_\psi) \Vdash \Diamond_j \neg(\psi_1 \wedge \ldots \wedge \psi_n)$ $(**)$ by soundness of Rule (*Forth–update*). Because Γ_1^α is a maximal consistent subset, there is $p_\psi \in \Gamma_1^\alpha$ such that $\psi \in \Gamma_1$, since by assumption $\mathcal{M}_c, \Gamma_1 \Vdash I^\alpha(\Gamma_1^\alpha)$. So, $\mathcal{M}_c, \Gamma \Vdash \Diamond_j(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \psi)$ and $\mathcal{M}_c, \Gamma^\alpha \Vdash \Diamond_j(\alpha_1 \wedge \ldots \wedge \alpha_n \wedge p_\psi)$. Therefore, by application of $(**)$, we obtain that $\mathcal{M}_c, \Gamma_f \Vdash \Diamond_j \neg(\psi_1 \wedge \ldots \wedge \psi_n)$, that is, $\mathcal{M}_c, \Gamma_f \Vdash \neg\Box_j(\psi_1 \wedge \ldots \wedge \psi_n)$. However, by definition of S_1, $\mathcal{M}_c, \Gamma_f \Vdash \Box_j \psi_1 \wedge \ldots \wedge \Box_j \psi_n$, and therefore $\mathcal{M}_c, \Gamma_f \Vdash \Box_j(\psi_1 \wedge \ldots \wedge \psi_n)$, which contradicts our last result. So, we reach a contradiction, therefore S is $\mathsf{L}_{\mathcal{R}} + (\textit{Forth–update})$–consistent. By Lemma 9, we can extend it to a maximal consistent subset Γ'_f of $\mathcal{L}_\mathcal{R}$. Finally, we show that (1) $\Gamma'_f \in \mathcal{R}_j(\Gamma_f)$ and (2) $(\Gamma_1, \Gamma_1^\alpha, \Gamma'_f) \in \mathcal{R}_c$. The first item follows from the definition of $\mathcal{R}_j$ and the fact that $S_1 \subseteq \Gamma'_f$. The second item follows from the definition of $\mathcal{R}_c$ and the fact that $S_2 \subseteq \Gamma'_f$. This concludes the proof. □

Note that our completeness results for (Atom–p), (Atom–$\neg p$), (Forth–update) and (Precondition) are at the level of *models* and not frames, unlike (Back–update):

Proposition 13 (Definability). *The $\mathcal{L}_\mathcal{R}$–sequent (*Back–update*) defines on the class of update* frames *the property (*Back–update*) of Definition 24.*

Proof. We have to show that for all update frames $\mathcal{F}$, we have that $\mathcal{F}$ satisfies (Back–update) iff $\mathcal{F}$ validates $\Box_j\varphi \circ \Box_j\alpha \vdash \Box_j(\varphi \circ \alpha)$. The left to right direction is routine, so we only prove the right to left direction. We reason by contraposition. Assume that an update frame $\mathcal{F} = (\mathcal{P}, \mathcal{R}_1, \ldots, \mathcal{R}_m, \mathcal{R})$ does not satisfy (Back–update). Then, there is $(w, e, w_f) \in \mathcal{R}$, there is $v_f \in \mathcal{R}_j(w_f)$ such that for all $v \in \mathcal{R}_j(w)$, all $f \in \mathcal{R}_j(e)$, $(v, f, v_f) \notin \mathcal{R}$. Then, we set an interpretation $\mathcal{I}$ on $\mathcal{F}$ such that $p \in \mathcal{I}(u)$ iff $u \in \mathcal{R}_j(w)$ and $p_\psi \in \mathcal{I}(u)$ iff $u \in \mathcal{R}_j(e)$ $(**)$ for a chosen $p \in ATM$ and a chosen $\psi \in \mathcal{L}$. Then, $(\mathcal{F}, \mathcal{I}), w_f \Vdash \Box_j p \circ \Box_j p_\psi$ because $(w, e, w_f) \in \mathcal{R}$. However, it holds that $(\mathcal{F}, \mathcal{I}), w_f \Vdash \Diamond_j \neg(p \circ p_\psi)$ because $v_f \in \mathcal{R}_j(w_f)$ and $(\mathcal{F}, \mathcal{I}), v_f \Vdash \neg(p \circ p_\psi)$ by condition $(**)$. Hence, $\mathcal{F}$ does not validate $\Box_j\varphi \circ \Box_j\alpha \vdash \Box_j(\varphi \circ \alpha)$, which concludes the first part of the proof. □

5.2 Characterization of the DEL Product Update

The following theorem shows that the DEL product update is entirely determined and characterized (modulo bisimulation) by the axioms and inference rules (*Atom–p & Atom–¬p*), (*Precondition*), (*Forth–update*), (*Back–update*):

Theorem 14 (Characterization Theorem). *Let $\mathcal{M}_\mathcal{R} = (\mathcal{P}, \mathcal{R}_1, \ldots, \mathcal{R}_m, \mathcal{R}, \mathcal{I})$ be an update model. Then,*

$\mathcal{M}_\mathcal{R}$ *validates (*Atom–p & Atom–$\neg p$*), (*Precondition*), (*Forth–update*), (*Back–update*)*

iff

$$((\mathcal{M}, w), (\mathcal{E}, e), (\mathcal{M}_f, w_f)) \in \mathcal{R} \textit{ iff } (\mathcal{M}_f, w_f) \leftrightarrows (\mathcal{M}, w) \otimes (\mathcal{E}, e).$$

Proof. The right to left direction is routine. To prove the left to right direction, we use the correspondence results of Theorem 12. One can easily show that the relation $Z \subseteq (\mathcal{M} \otimes \mathcal{E}) \times \mathcal{M}_f$ defined as follows is a bisimulation relation:

$(\mathcal{M}, w) \otimes (\mathcal{E}, e) Z (\mathcal{M}_f, w_f)$ iff $((\mathcal{M}, w), (\mathcal{E}, e), (\mathcal{M}_f, w_f)) \in \mathcal{R}$ □

As our denomination suggests, the conditions (*Back–update*) (together with (*Precondition*)) and (*Forth–update*) somehow encode respectively the back and forth clauses of bisimulation; the conditions (*Atom–p & Atom–¬p*) somehow encode the Atom clause of bisimulation. This result generalizes the correspondence result for public announcement logic (van Benthem, 2007a) and solves an open problem raised by van Benthem (2011a). Note that for the specific case of public announcement, we also had conditions resembling the back and forth conditions of the definition of bisimulation. This leads us to define the notion of a DEL product update model:

Definition 25 (DEL product update model)**.** A *DEL product update model* is an update model $\mathcal{M}_\otimes = (\mathcal{P}, \mathcal{R}_1, \ldots, \mathcal{R}_m, \mathcal{R}_\otimes, \mathcal{I})$ such that:

- $\mathcal{P} := (P, =)$ where $P \subseteq \mathcal{C} \cup \mathcal{C}^\alpha$;
- $\mathcal{R}_j \subseteq \mathcal{P} \times \mathcal{P}$ is a positive two-place accessibility relation on $\mathcal{P}$ for each $j \in AGT$ such that for all $x, y \in \mathcal{P}$, where $x = (\mathcal{M}_x, w_x)$ and $y = (\mathcal{M}_y, w_y)$:

$$x \in \mathcal{R}_j(y) \ \text{ iff } \ \mathcal{M}_x = \mathcal{M}_y \text{ and } w_x \in R_j(w_y)$$

- $\mathcal{R}_\otimes := \{(x, y, z) \in \mathcal{C} \times \mathcal{C}^\alpha \times \mathcal{C} : x \otimes y \leftrightarroweq z\}$ is a plump accessibility relation on $\mathcal{P}$;
- $\mathcal{I}(x) := I(x)$, for all $x \in \mathcal{C} \cup \mathcal{C}^\alpha$. □

The DEL product update model is a $\mathcal{L}_{\mathsf{Sub}}$–model where points are pointed $\mathcal{L}$–models and pointed $\mathcal{L}_\alpha$–models. The ternary relation $\mathcal{R}_\otimes$ is defined and motivated by the explanations of the previous section. Note that the accessibility relations R_j of $\mathcal{L}$–models and $\mathcal{L}_\alpha$–models are seen in this definition as positive two-place accessibility relations $\mathcal{R}_j$.

Fact 6. *Any DEL product update model is an update model.*

Proof. It follows straightforward from the definitions of update model and DEL product update model. □

Fact 7. *The ternary relation $\mathcal{R}_\otimes$ induced by the DEL product update is a plump accessibility relation on the point set $\mathcal{P} = (\mathcal{C} \cup \mathcal{C}^\alpha, \leftrightarroweq)$. That is, for all $(\mathcal{M}, w), (\mathcal{M}', w') \in \mathcal{C}$, all $(\mathcal{E}, e), (\mathcal{E}', e') \in \mathcal{C}^\alpha$ and all $(\mathcal{M}_f, w_f), (\mathcal{M}'_f, w'_f) \in \mathcal{C}$ such that $((\mathcal{M}, w), (\mathcal{E}, e), (\mathcal{M}_f, w_f)) \in \mathcal{R}$,*

$$\begin{aligned} &\textit{if } (\mathcal{M}, w) \leftrightarroweq (\mathcal{M}', w'),\ (\mathcal{E}, e) \leftrightarroweq (\mathcal{E}', e') \textit{ and } (\mathcal{M}_f, w_f) \leftrightarroweq (\mathcal{M}'_f, w'_f), \\ &\textit{then } ((\mathcal{M}', w'), (\mathcal{E}', e'), (\mathcal{M}'_f, w'_f)) \in \mathcal{R}. \end{aligned} \tag{23}$$

Proof. It follows straightforwardly from the definition of the product update. □

So, applying the DEL product update with bisimilar $\mathcal{L}_\alpha$–models yields the same results (modulo bisimulation). But the converse turns out to be false: $\mathcal{L}_\alpha$–models may have the same update effects without being bisimilar. In fact, there is a weaker notion than bisimulation for $\mathcal{L}_\alpha$–models, called *emulation* (Eijck et al., 2012), that still fulfills Expression (23).

5.3 A Gentzen Calculus for DEL

Putting all our results together, we obtain an axiomatization of the DEL product update which is different from the axiomatization of (Aucher, 2011). One of its advantage is its modularity. Indeed, we do not need to resort to an *external* calculus to take into account the base epistemic or event logic: this base calculus is already present in the axiomatization at the same level than the other connectives of the language, and this base calculus can also be modified or enriched.

Theorem 15 (Soundness and Completeness)**.** *Let* $\mathsf{A} = \{$*(*Atom–p & Atom–$\neg p$*), (*Precondition*), (*Forth–update*), (*Back–update*)*$\}$*. Then, the cut-free sequent calculus* $\mathsf{L}_\otimes := \mathsf{L}_\mathcal{R} + \mathsf{A}$ *is sound and complete for* $\mathcal{L}_\mathcal{R}$ *w.r.t. the class of DEL product update models.*

Proof. The soundness is routine. The completeness proof is similar to Theorem 7 and ultimately relies on Theorem 14. □

5.4 DEL Operators are Substructural Operators

In this section, we will show that the DEL operators introduced in (Aucher, 2011, 2012) correspond to the substructural operators $\circ, \supset$ and $\subset$. We will also relate the work of van Benthem on dynamic inference with the DEL–sequents of (Aucher, 2011, 2012; Aucher et al., 2012).

5.4.1 Dynamic Inferences and DEL–sequents

Dynamic Inferences In the so-called 'dynamic turn', van Benthem was interested in various dynamic styles of inference where propositions are procedures changing information states. These dynamic styles of inference differ greatly from the classical Tarskian's valid inferences because the latter are supposed to transmit and preserve truth. Among various dynamic styles of inference (such as the so-called test-test, update-update or update-test consequence (van Benthem, 1991, 1996; Muskens et al., 2011)), he studied the concrete following one, which can be defined within the DEL framework:

Definition 26 (Dynamic inference, van Benthem 2003)**.** Let $\varphi_0, \varphi_1, \ldots, \varphi_n, \psi \in \mathcal{L}$. We define the *dynamic inference* $\varphi_0, \varphi_1, \ldots, \varphi_n \models \psi$ as follows:

$\varphi_1, \ldots, \varphi_n \models \varphi$ iff for all pointed $\mathcal{L}$–model $(\mathcal{M}, w)$, and public announcement $\mathcal{L}_\alpha$–models $(\mathcal{E}_1, e_1), \ldots, (\mathcal{E}_n, e_n)$ of $\varphi_1, \ldots, \varphi_n$ respectively, $(\mathcal{M}, w) \otimes (\mathcal{E}_1, e_1) \otimes \ldots \otimes (\mathcal{E}_n, e_n) \models \varphi$.

□

van Benthem noticed that various dynamic styles of inference obey structural rules of inference which are non-classical. For example, all the structural rules of classical logic of Figure 7 fail for dynamic inference, but the structural rules below characterize completely the dynamic inference (van Benthem, 2003) (below, $\overrightarrow{\varphi}$ stands for $\varphi_1, \ldots, \varphi_n$ and $\overrightarrow{\psi}$ stands for $\psi_1, \ldots, \psi_n$, where $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in \mathcal{L}$):

$$\text{if } \overrightarrow{\varphi} \models \varphi \text{ then } \psi, \overrightarrow{\varphi} \models \varphi \qquad \text{(Left-Monotonicity)}$$
$$\text{if } \overrightarrow{\varphi} \models \varphi \text{ and } \overrightarrow{\varphi}, \varphi, \overrightarrow{\psi} \models \psi \text{ then } \overrightarrow{\varphi}, \overrightarrow{\psi} \models \psi \qquad \text{(Left-Cut)}$$
$$\text{if } \overrightarrow{\varphi} \models \varphi \text{ and } \overrightarrow{\varphi}, \overrightarrow{\psi} \models \psi \text{ then } \overrightarrow{\varphi}, \varphi, \overrightarrow{\psi} \models \psi \qquad \text{(Cautious Monotonicity)}$$

DEL–sequents In (Aucher, 2011), I introduced what I called DEL–sequents. They are a particular sort of dynamic inference and are defined as follows:

Definition 27 (DEL–sequent, Aucher 2011)**.** Let $\varphi, \varphi_f \in \mathcal{L}$ and $\alpha \in \mathcal{L}_\alpha$. We define the logical consequence relation $\varphi, \alpha \models \varphi_f$ as follows:

$\varphi, \alpha \models \varphi_f$ iff for all pointed $\mathcal{L}$–model $(\mathcal{M}, w)$, all $\mathcal{L}_\alpha$–model $(\mathcal{E}, e)$ such that $\mathcal{M}, w \models I^\alpha(e)$, $\mathcal{M}, w \models \varphi$ and $\mathcal{E}, e \models \alpha$, it holds that $(\mathcal{M}, w) \otimes (\mathcal{E}, e) \models \varphi_f$.

□

In (Aucher et al., 2012), DEL–sequents are generalized to take into account sequences of events and not only 'one-shot' occurrence of events. Several generalized DEL–sequents are introduced in (Aucher et al., 2012) but they are all reducible to the following one:

Definition 28 (Generalized DEL–sequent, Aucher et al. 2012)**.** Let $\varphi_0, \ldots, \varphi_n \in \mathcal{L}$, let $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_\alpha$ and let $\psi \in \mathcal{L}$. Then,

$$\varphi_0, \alpha_1, \varphi_1, \ldots, \alpha_n, \varphi_n \models \psi$$
iff
if for all pointed $\mathcal{L}$–model $(\mathcal{M}, w)$, and $\mathcal{L}_\alpha$–models $(\mathcal{E}_1, e_1), \ldots, (\mathcal{E}_n, e_n)$ such that for all $i \in \{1, \ldots, n\}$, $\mathcal{E}_i, e_i \models \alpha_i$, $(\mathcal{M}, w) \otimes (\mathcal{E}_1, e_1) \otimes \ldots \otimes (\mathcal{E}_i, e_i)$ is defined and makes φ_i true, then it holds that $(\mathcal{M}, w) \otimes (\mathcal{E}_1, e_1) \otimes \ldots \otimes (\mathcal{E}_n, e_n) \models \psi$.

□

As one can easily notice, dynamic inferences can be translated into DEL–sequents if we resort to the common knowledge/belief operator $\Box^*_{AGT}\varphi$ (see for example (Fagin et al., 1995) for a definition and a detailed study of this operator):

Proposition 16. *Let $\varphi_0, \varphi_1, \ldots, \varphi_n, \varphi \in \mathcal{L}$. Then, the following holds:*

$$\varphi_1, \ldots, \varphi_n \models \varphi \quad \textit{iff} \quad \top, p_{\varphi_1} \land \Box^*_{AGT} p_{\varphi_1}, \ldots, \top, p_{\varphi_n} \land \Box^*_{AGT} p_{\varphi_n}, \top \models \varphi \land \Box^*_{AGT}\varphi$$

Thus, DEL–sequents are more expressive than dynamic inferences, and also more abstract because they 'operate' at a deeper level, a semantical one. It is this more general and abstract approach towards dynamic styles of inference that will allow us to relate more precisely and closely DEL with substructural logics, and explain to a certain extent why the substructural phenomena occurring in dynamic inferences and observed by van Benthem arise.

5.4.2 DEL–sequents for Progression, Regression and Epistemic Planning

Recently again, van Benthem (2010) expressed some worries about interpreting the Lambek Calculus (the paradigmatic substructural logic) as a base logic of information flow while trying to connect the operators $\circ, \supset$ and $\subset$ of substructural logic to some sort of DEL operators. Indeed, the DEL operators usually rely on the regular algebra of sequential composition, choice and iteration which are of a quite different nature. Recently, I introduced some DEL operators called progression, regression and epistemic planning (Aucher, 2011, 2012), the operator of regression being a natural generalization of the standard and original action modality $[\mathcal{E}, e]\varphi$ of DEL (Batlag et al., 1998). It turns out that these operators can all be identified with connectives of the substructural language $\mathcal{L}_{\mathcal{R}}$. After some informal motivations, we first briefly recall their definitions and then we give our correspondence results between the two kinds of operators.

As spelled out in Section 2, the core idea of DEL is to split the task of representing the agents' beliefs into three parts. Consequently, within the logical framework of DEL, one can express uniformly epistemic statements about:

(i) what is true about an initial situation,

(ii) what is true about an event occurring in this situation,

(iii) what is true about the resulting situation after the event has occurred.

From a logical point of view, this trichotomy begs the following three questions (which were already raised by Kooi (2007)). In these questions, φ, α and φ_f are three epistemic formulas describing respectively (i), (ii) and (iii).

- **Question 1**:
  1. Given (i) and (ii), what can we infer about (iii): $\varphi, \alpha \models \varphi_f$?
  2. How can we build a single formula $\varphi \otimes \alpha$ which captures all the information which can be inferred about (iii) from φ and α?
- **Question 2**:
  1. Given (i) and (iii), what can we infer about (ii): $\varphi, \varphi_f \models \alpha$?
  2. How can we build a single formula $\varphi \otimes_P \varphi_f$ which captures all the information which can be inferred about (ii) from φ and φ_f?
- **Question 3**:
  1. Given (ii) and (iii), what can we infer about (i): $\alpha, \varphi_f \models \varphi$?
  2. How can we build a single formula $\varphi \oslash \varphi_f$ which captures all the information which can be inferred about (i) from α and φ_f?

Providing formal tools that answer these questions leads to applications in artificial intelligence and theoretical computer science, and as it turns out, some of them have already been addressed in DEL and other logical formalisms.

- **Question 1: Progression.** Answering the first question leads to the development of tools that can be used by (artificial) agents to compute autonomously their representation of situations as events occur or to reason about the effects of these events. This question has been addressed in the situation calculus, where it is related to the notion of *progression* (Reiter, 2001). In the logics of programs, our DEL-sequent $\varphi, \alpha \models \varphi_f$ correspond to the partial correctness specifications $\{\varphi\}\pi\{\varphi_f\}$ of Hoare's logic (1969) which read as "after every successful execution of program π starting from a state where precondition φ holds, postcondition φ_f holds in the final state". Likewise, our formula $\varphi \otimes \alpha$ corresponds to the strongest post-condition of Propositional Dynamic Logic Pratt (1976). That the product update of DEL is in fact the same as the strongest post-condition has been elaborated on and proved in an algebraic setting in Baltag et al. (2005). A sequent calculus is also provided in this algebraic setting.

- **Question 2: Epistemic planning.** Answering the second question also leads to applications in artificial intelligence in the area of *epistemic planning*: (artificial) agents often need to determine autonomously which actions they need to perform in order to achieve a given epistemic goal. This second question is also related to the notion of explanation and has been dealt with in the event calculus of Shanahan (1997) for instance, where it is shown that planning problems can be handled via abduction (using logic programming). In computer science, this second question is also related to the *synthesis problem* raised by Church (1957) in its full generality. He asked whether, given a desired relation between a set of inputs and a set of outputs, we can construct a function that produces the desired outputs from arbitrary inputs. This problem has been declined as the problem of program synthesis: given a specification, can we construct a program that is guaranteed to satisfy this specification? It was extensively studied in the 1980s and 1990s for temporal logic specifications. The synthesis problem is more challenging when the input is incomplete (Kupferman and Vardi, 1999). Open (reactive) environments can be a reason of incompleteness of the input, and epistemic logic is a natural formalism to resort to model such situations, as argued by Halpern and Moses (1990). For single-agent temporal epistemic logic, this synthesis problem has been solved by van der Meyden and Vardi (1998). However, this problem has not been addressed so far within the DEL approach, although its methodology and formal setting lend itself rather naturally to address it.

- **Question 3: Regression.** Answering the third question is related to the notion of *regression* introduced in the situation calculus (Reiter, 2001). This technique is used to determine whether a statement holds after a sequence of events (called the *projection* problem) by reducing (regressing) this statement about the resulting situation to a statement about the initial situation. In DEL, regression corresponds to the classical reduction method used to prove completeness of an axiomatization: a formula with dynamic operator(s) is 'reduced' equivalently to a formula without dynamic operator by pushing the dynamic operator through the logical connectives, performing some kind of regression of the initial formula with dynamic operator. In (Aucher, 2012), our inductive definition of the regression of φ_f by α, i.e. $\alpha \oslash \varphi_f$, is based on the reduction axioms of DEL (Baltag and Moss, 2004). Note that in Propositional Dynamic Logic, $\neg(\varphi \oslash \neg\varphi_f)$ also corresponds to the weakest precondition.

Now, we provide the formal definitions of these operators of progression, regression and epistemic planning.

Progression The operator of *progression* is denoted $\otimes$ in (Aucher, 2011). In (Aucher, 2012, Def. 41), a constructive definition of this operator is provided using characteristic formulas (called "Kit Fine" formulas). Here, we provide an alternative and non–constructive definition of the *progression* of φ by α, denoted $\varphi \otimes \alpha$:

Theorem 17. *Let $(\mathcal{M}_f, w_f)$ be a pointed $\mathcal{L}$–model and let $\varphi \in \mathcal{L}$ and $\alpha \in \mathcal{L}_\alpha$. Then,*

$$\mathcal{M}_f, w_f \models \varphi \otimes \alpha \quad \textit{iff} \quad \begin{array}{l}\textit{there is a pointed } \mathcal{L}\textit{–model } (\mathcal{M}, w) \textit{ and a pointed} \\ \mathcal{L}_\alpha\textit{–model } (\mathcal{E}, e) \textit{ such that } (\mathcal{M}, w) \otimes (\mathcal{E}, e) \leftrightarrows (\mathcal{M}_f, w_f), \\ \mathcal{M}, w \models \varphi \textit{ and } \mathcal{E}, e \models \alpha\end{array}$$

Proof. It follows from Lemmata 43 and 44 of (Aucher, 2011). □

Epistemic Planning The operator of *epistemic planning* is denoted $\oslash_P$ in (Aucher, 2012). It is defined relatively to a finite set P of formulas/preconditions/atomic events. In (Aucher, 2012, Def. 14–15), a constructive definition of this operator is provided using characteristic formulas (called "Kit Fine" formulas). As it turns out, an alternative and non–constructive definition of the *epistemic planning from φ to φ_f*, denoted $\varphi \oslash_P \varphi_f$, exists as well:

Theorem 18 (Aucher 2012)**.** *Let $\varphi, \varphi_f \in \mathcal{L}$ and let P be a finite subset of $\mathcal{L}$. Then, for all P–complete $\mathcal{L}_\alpha$–model $(\mathcal{E}, e)$, it holds that*

$$\mathcal{E}, e \models \varphi \oslash_P \varphi_f \quad \textit{iff} \quad \begin{array}{l}\textit{there is } (\mathcal{M}, w) \textit{ such that } \mathcal{M}, w \models \varphi, \\ \mathcal{M}, w \models I^\alpha(e) \textit{ and } (\mathcal{M}, w) \otimes (\mathcal{E}, e) \models \varphi_f\end{array}$$

The dual of the operator $\varphi \oslash_P \varphi_f$ is defined by:

$$\varphi[\oslash]_P\varphi_f := \neg(\varphi \oslash_P \neg\varphi_f) \tag{24}$$

Theorem 18 entails that $\varphi[\oslash]_P\varphi_f$ can be alternatively defined as follows: for all P–complete $\mathcal{L}_\alpha$–model $(\mathcal{E}, e)$, it holds that

$$\mathcal{E}, e \models \varphi[\oslash]_P\varphi_f \quad \text{iff} \quad \begin{array}{l}\text{for all } (\mathcal{M}, w) \text{ such that } \mathcal{M}, w \models \varphi, \text{ if} \\ \mathcal{M}, w \models I^\alpha(e) \text{ then } (\mathcal{M}, w) \otimes (\mathcal{E}, e) \models \varphi_f\end{array} \tag{25}$$

Example 4. In the situation depicted in the $\mathcal{L}$-model of Figure 1, agent B does not know that agent A has the red card and does not know that agent C has the blue card: $\mathcal{M}, w \models (\Diamond_B r_A \wedge \Diamond_B \neg r_A) \wedge (\Diamond_B b_C \wedge \Diamond_B \neg b_C)$. Our problem is therefore the following:

> What sufficient and necessary property (i.e. 'minimal' property) an event should fulfill so that its occurence in the initial situation $(\mathcal{M}, w)$ results in a situation where agent B knows the true state of the world, i.e. agent B knows that agent A has the red card and that agent C has the blue card?

The answer to this question obviously depends on the kind of atomic events we consider. In this example, the events $P = \{p_{b_C}, p_{r_A}, p_{w_B}\}$ under consideration are the following. First, agent C shows her blue card (p_{b_C}), second, agent A shows her red card (p_{r_A}), and third, agent B herself shows her white card (p_{w_B}). Answering this question amounts to compute the formula $(M, w) \oslash_P \Box_B (r_A \wedge b_C \wedge w_B)$. Applying the algorithm of (Aucher, 2012, Definition 15), we obtain that

$$(\mathcal{M}, w) \oslash_P \Box_B (r_A \wedge b_C \wedge w_B) \leftrightarrow \Box_B (p_{b_C} \vee p_{r_A}) \text{ is valid.}$$

In other words, this result states that agent B should believe either that agent A shows her red card or that agent C shows her blue card in order to know the true state of the world. Indeed, since there are only three different cards which are known by the agents and agent B already knows her card, if she learns the card of (at least) one of the other agents, she will also be able to infer the card of the third agent. □

Regression The operator of *regression* is denoted $\oslash$ in (Aucher, 2011). In (Aucher, 2012, Def. 41), a constructive definition of this operator is provided using characteristic formulas (called "Kit Fine" formulas) by adapting and translating the reduction axioms of (Batlag et al., 1998). As it turns out, an alternative and non–constructive definition of the *regression* of φ_f by α, denoted $\alpha \oslash \varphi_f$, exists as well:

Theorem 19 (Aucher 2012)**.** *Let $\alpha \in \mathcal{L}_\alpha$ and $\varphi_f \in \mathcal{L}$. Then, for all $\mathcal{L}$-model $(\mathcal{M}, w)$, it holds that*

$$\mathcal{M}, w \models \alpha \oslash \varphi_f \quad \textit{iff} \quad \begin{array}{l} \textit{there is } (\mathcal{E}, e) \textit{ such that } \mathcal{E}, e \models \alpha, \\ \mathcal{M}, w \models I^\alpha(e) \textit{ and } (\mathcal{M}, w) \otimes (\mathcal{E}, e) \models \varphi_f \end{array}$$

Note that we could define a dual operator of $\alpha \oslash \varphi_f$ as follows:

$$\alpha[\oslash]\varphi_f = \neg (\alpha \oslash \neg\varphi_f) \tag{26}$$

Then, the counterpart of Theorem 19 for this dual operator is as follows:

$$\mathcal{M}, w \models \alpha[\oslash]\varphi_f \quad \text{iff} \quad \begin{array}{l} \text{for all } (\mathcal{E}, e) \text{ such that } \mathcal{E}, e \models \alpha, \\ \text{if } \mathcal{M}, w \models I^\alpha(e) \text{ then } (\mathcal{M}, w) \otimes (\mathcal{E}, e) \models \varphi_f \end{array} \tag{27}$$

As shown in (Aucher, 2012, Sec. 6), the operator $\alpha[\oslash]\varphi_f$ is a generalization of the original and more standard DEL operator $[\mathcal{E}, e]\varphi$ almost exclusively used in the DEL literature (Batlag et al., 1998).

Substructural operators	DEL operators
$\circ$	$\otimes$
$\supset$	$[\oslash]$
$\subset$	$[\obslash]$

Figure 9: Correspondence between DEL and Substructural Operators

Correspondence between DEL and Substructural Operators As one can easily notice, there is a strong similarity between the operations of progression, epistemic planning and regression and the operations of substructural logic, more precisely of the Lambek Calculus. In fact, there exists a rigorous mapping between them, as the following theorem shows:

Theorem 20. *Let $\mathcal{M}_\otimes$ be the DEL product update model where the point set is $\mathcal{P} := (\mathcal{C} \cup \mathcal{C}^\alpha, \Leftrightarrow)$. Let P be a finite subset of $\mathcal{L}$, let $x = (\mathcal{M}, w) \in \mathcal{C}$ and let $y = (\mathcal{E}, e) \in \mathcal{C}^\alpha_P$ be a P-complete pointed event model. Let $\varphi, \psi \in \mathcal{L}$ and let $\alpha \in \mathcal{L}_\alpha$. Then,*

$$\begin{array}{lll} \mathcal{M}_\otimes, x \Vdash \varphi \circ \alpha & \textit{iff} & \mathcal{M}, w \models \varphi \otimes \alpha \\ \mathcal{M}_\otimes, x \Vdash \alpha \supset \varphi & \textit{iff} & \mathcal{M}, w \models \alpha[\oslash]\varphi \\ \mathcal{M}_\otimes, y \Vdash \psi \subset \varphi & \textit{iff} & \mathcal{E}, e \models \varphi[\obslash]_P\psi \end{array}$$

Moreover, for all $\alpha, \alpha_1, \ldots, \alpha_n \in \mathcal{L}_\alpha$, for all $\varphi, \psi, \varphi_0, \varphi_1, \ldots, \varphi_n \in \mathcal{L}$, we have:

$$\begin{array}{lll} \varphi; \alpha \Vdash \psi & \textit{iff} & \varphi, \alpha \models \psi \\ (((\varphi_0; \alpha_1), \varphi_1); \ldots; \alpha_n), \varphi_n \Vdash \psi & \textit{iff} & \varphi_0, \alpha_1, \varphi_1, \ldots, \alpha_n, \varphi_n \models \psi \end{array}$$

Proof. It follows straightforwardly from Theorems 17, 18, 19 and the truth conditions of the operators $\circ, \supset, \subset$ and ;. □

Theorem 20 explains to a certain extent *why* some substructural phenomena arise in the dynamic inferences of Section 5.4.1. As observed by van Benthem, "it seemed that structural rules address mere *symptoms* of some underlying phenomenon" (van Benthem, 2011a, p. 297). I claim that these "symptoms" are caused at a deeper semantic level by the fact that an update, and in that case the DEL product update, can be represented by the ternary relation of substructural logics.

The key Theorem 42 of (Aucher, 2011) relates DEL–sequents and the operator of progression: for all $\varphi, \varphi_f \in \mathcal{L}$ and $\alpha \in \mathcal{L}_\alpha$, it holds that

$$\varphi, \alpha \models \varphi_f \text{ iff } \varphi \otimes \alpha \models \varphi_f. \tag{28}$$

As it turns out, this theorem is also valid in any substructural logic: it corresponds to the theorem of Expression (2). More generally, all the theorems of the non-associative Lambek calculus hold in our DEL setting if we use the translation given in Figure 9. In particular, we have the following results which are the counterparts of Expressions (3), (4) and (5) in our setting:

Corollary 1. *Let P be a finite subset of $\mathcal{L}$. For all $\varphi, \varphi_f \in \mathcal{L}$ and $\alpha \in \mathcal{L}_\alpha$, it holds that*

$$\varphi; \alpha \Vdash \varphi_f \quad \textit{iff} \quad \varphi \Vdash \alpha[\oslash]\varphi_f \tag{29}$$

$$\varphi \Vdash \alpha[\oslash]\varphi_f \quad \textit{iff} \quad \varphi \otimes \alpha \Vdash \varphi_f \tag{30}$$

$$\varphi \otimes \alpha \Vdash \varphi_f \quad \textit{iff} \quad \alpha \Vdash_{C_P^\alpha} \varphi[\oslash]_P \varphi_f \tag{31}$$

$$\varphi \Vdash \alpha[\oslash]\varphi_f \quad \textit{iff} \quad \alpha \Vdash_{C_P^\alpha} \varphi[\oslash]_P \varphi_f \tag{32}$$

6 Conclusion

We introduced a logic called *update logic* where updates are the central objects of study. We elicited a number of axioms and inference rules that define specific properties of updates. Even if this paves the way for a correspondence theory analyzing and studying the notion of update, it remains to characterize other important properties of updates in terms of axioms and inference rules, such as for instance determinism or bisimulation invariance. In fact , we view our contributions as a first step in our exploration of the update universe. Indeed, we focused our attention in this report on the DEL product update. It is, however, a particular kind of update operator and the ternary relation of substructural logics could actually be a representation of any sort of update. In that respect, in the DEL paradigm, Liu (2008) mentions some of the alternatives to the DEL product update and van Eijck et al. (2011) recently introduced a new sort of product update (subsequently studied by Aceto et al. (2013)). But more generally, the ternary relation could also represent the various revision and update operators which have been studied in the logics of "common sense reasoning" of artificial intelligence and philosophical logic, such as conditional logic (Nute and Cross, 2001), default and non-monotonic logics (Makinson, 2005; Gabbay et al., 1998), belief revision theory (Gärdenfors, 1988), etc. . . In fact, in the DEL framework, numerous product update rules for belief revision have been proposed in a setting with more refined representations of uncertainty (by means of plausibility or probability measures for instance): (Aucher, 2004; van Ditmarsch, 2005; Aucher, 2007; Baltag and Smets, 2006, 2008a; van Benthem, 2007a; Liu, 2008; van Benthem et al., 2009b). Nevertheless, one still needs to show that the proof techniques developed in this report can be adapted to these more refined representations of uncertainty.

Moreover, we proposed Gentzen calculi for our update logic and for DEL, using our correspondence results. To prove the completeness of these calculi, we introduced new proof techniques based on a Henkin construction. These techniques are not specific to our update logic or DEL and we applied them to modal logic (and propositional logic).

To conclude, we have also shown that DEL can be embedded within the framework of substructural logic in an intuitively meaningful way, in the sense that in this embedding the intuitions underlying the DEL framework provide a reasonable and meaningful interpretation of the ternary semantics of substructural logic. In addition to other non-classical logics such as the Lambek calculus, linear logic, relevance logic, arrow logic,. . . this new embedding illustrates the richness and expressiveness of the general framework of substructural logics: it is defined in such a way that it can even capture in a meaningful way logics which are sometimes considered as isolated or 'exotic', like DEL. Therefore, our results are new evidences in support to the fact that the framework of substructural logics can indeed be considered as an unifying framework for non-classical logics.

Acknowledgements I thank Olivier Roy and Ole Hjortland for organizing and inviting me to an inspiring workshop on substructural epistemic logic in Munich in February 2013. Also, I thank Johan van Benthem for comments on an earlier version of this report.

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ISSN 0249-6399